Continuum Branching Observable in Higher Genus

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Abstract

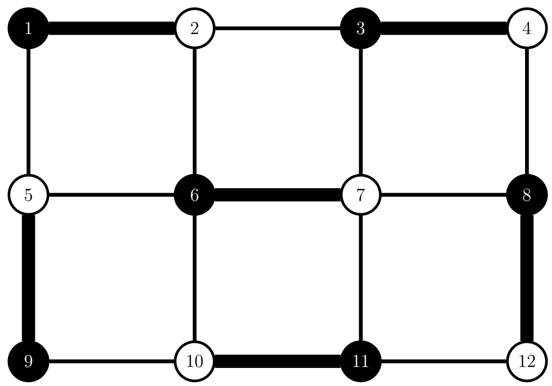
For all fixed sufficiently large genus $g = 0, 1, \geqslant 2$, multiedge connected, dual graph, we give a uniform bipartite observable of Grassmann kernel transfer matrices. On special hexagonal domain, we prove discriminant steepest descent of Grassmann kernel logarithmic asymptotics, and free Dirac Fermion convergence $\Psi_{12} \times (1 + \mathcal{O}(1))$. We conjecture: In large deviation functional, the Green's function $\mathscr{G}$ for Dirichlet problem of variational principle minimizer is observable in the kernel asymptotics.

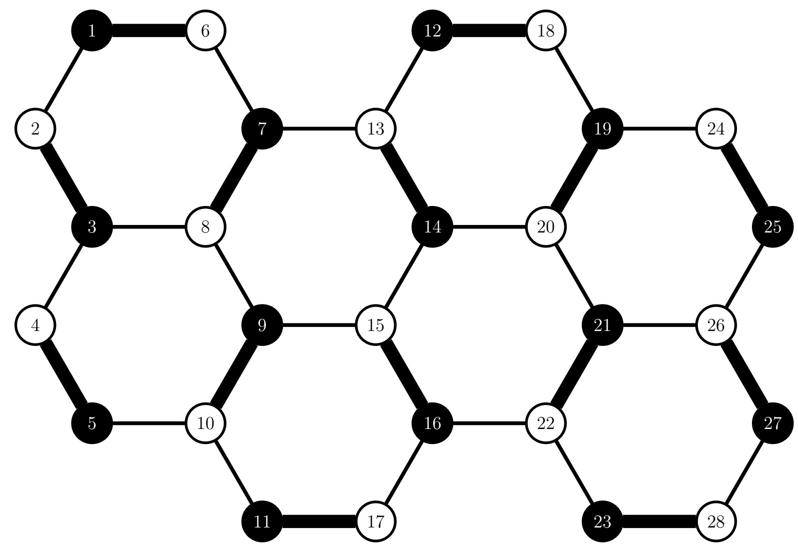
Keywords: Continuum-branching, higher-genus, observable

1 Characterizations

Bipartite implies no adjacent-black (-white) vertices for all $\mathscr{V}_X = \mathscr{V}_X^{\bullet} \sqcup \mathscr{V}_X^{\circ}$:
$\mathscr{V}_X^{\bullet} = \big(\bullet_{k_\xi} \cap D : \bigcap_{\xi \neq \eta} \{k_\xi, k_\eta, D\} = \emptyset \mid D = \big\{\xi = \{\bullet_{k_\xi}, \circ_{\ell_\xi}\}_{k \neq \ell};\ |k_\xi \cap D| = 1\big\}\big)$.

Instance.





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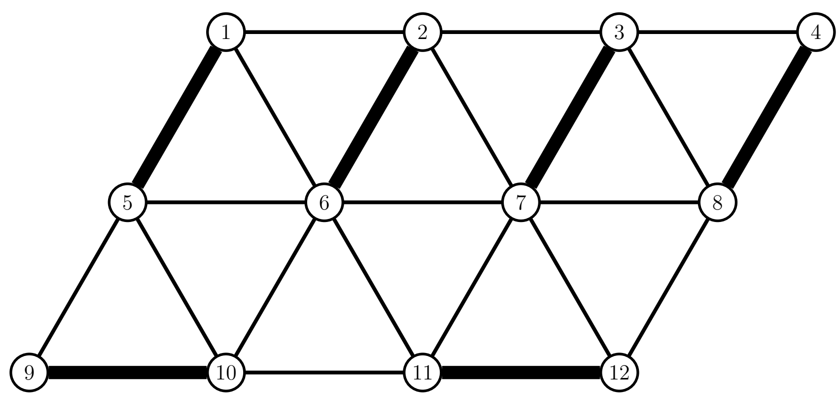
Non-instance.



$\left(\begin{array}{c}\textit{no bipartite structure}\\ \textit{on triangular grids}\end{array}\right)$.

Derivation. $\mathbb{R}^n$ r.v. $X=(X_{ab}; a<b\in\mathscr{V}_X)$ is Gaussian G iff $X=\mu$ a.s., for

$$\mathbb{E}[\exp(it'X)] = \exp\left(it'\mu - \frac{1}{2}t'\Sigma t\right) \;\Big|\; \mu_{ab} = \mathbb{E}[X_{ab}],\ \Sigma_{ab,\alpha\beta} = \mathsf{cov}[X_{ab}, X_{\alpha\beta}]$$

$$\iff \mathbb{P}\{X \in dx\} = \frac{1}{\sqrt{(2\pi)^n \det\Sigma}} \exp\left(\frac{-(X-\mu)^T\,\Sigma^{-1}\,(X-\mu)}{2}\right)dx$$

iff $t'X=\sum_{a<b\,:\,a\sim b} t_{ab}X_{ab}$ is $\mathbb{R}$-G, $\forall\, t\in\mathbb{R}^n$ where X is $\mathbb{R}$-G iff $X=\mu$ a.s., for

$$\mathbb{E}[\exp(itX)] = \exp\left(i\mu t - \frac{\Sigma t^2}{2}\right) \;\Big|\; \mu = \mathbb{E}[X],\ \Sigma = \mathsf{var}[X],\ \forall\, t \in \mathbb{R}$$

$$\iff \mathbb{P}\{X \in dx\} = \frac{1}{\sqrt{2\pi\Sigma}} \exp\left(-\frac{(X-\mu)^2}{2\Sigma}\right)dx$$

where X is independent iff $\Sigma_{ab,\alpha\beta} = 0$, $\forall\, X_{ab}, X_{\alpha\beta} \mid \{a,b\} \neq \{\alpha,\beta\}$;
and, X is absolutely continuous iff Σ is non-singular.

Derivation. $UX \overset{\mathsf{d}}{=} X$, for all $U^T U = UU^T = I$,
iff centered $X \in \mathbb{R}^n$ and Hermitian $H \mid U = e^{\sqrt{-1}H}$;
iff $\frac{X}{\|X\|} = \frac{X}{\left(\sum\limits_{\xi=1}^{n} X^2_{\sigma_{2\xi-1}\sigma_{2\xi}}\right)^{1/2}}$ is uniformly distributed on $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$
where X is standard iff centered ($\mu = \overline{0}$) and $\Sigma = I$, i.e.$\forall\, X_{\sigma_{2\xi-1}\sigma_{2\xi}} \sim \mathcal{N}(0,1)$.

Derivation. For n centered Gaussian, resp. Maxwell, particle velocity X,

$$\mathbb{E}[itX] = \prod_{\xi=1}^{n} \Phi(t_\xi) \;=\; \Phi\left(\sqrt{t_1^2 + \cdots + t_n^2}\right) \;\; \Big| \; \Phi(t) = \exp\left(-\frac{1}{2}\Sigma t^2\right)$$

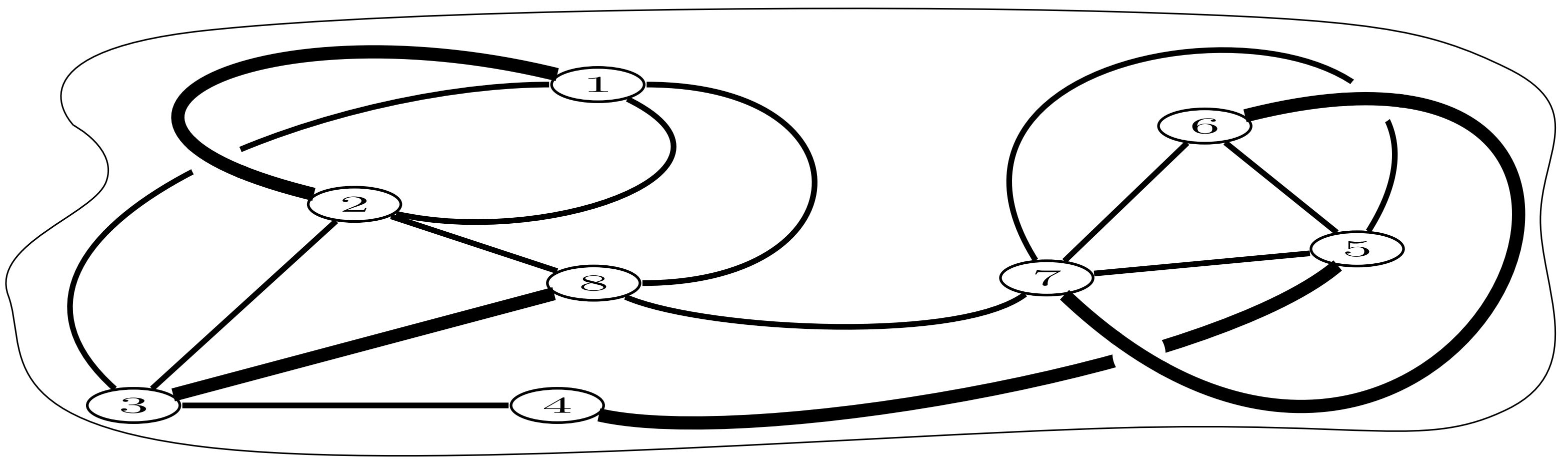
resp. $\mathbb{E}[itX] = \prod\limits_{\xi=1}^{3} \Phi(t_\xi) = \Phi\left(\sqrt{t_1^2 + t_2^2 + t_3^2}\right) \Big| \; \Phi(t) = \exp\left(-\frac{1}{2}\Sigma t^2\right), \Sigma \geqslant 0.$

Remark. For $n \longrightarrow +\infty$, on support S, continuous centered density $f(x)$:

$$-\frac{1}{n} \ln f^{\otimes n}(X_{\sigma_1\sigma_2}, \ldots, X_{\sigma_{2n-1}\sigma_{2n}}) \xrightarrow{\text{a.s.}} \mathbb{E}\left[-\ln f(X)\right] \;=\; -\int_S f(x) \ln f(x) \;=\; \frac{1}{2} \ln(\det(2\pi e \Sigma)).$$

1.1 Partition

Definition. Embedding $X\subset\overline{\mathcal{M}}_g\,|\,\mathcal{V}_X=\big(k_\xi\cap D:\bigcap_{\xi\neq\eta}\{k_\xi,k_\eta,D\}=\emptyset\big)$ is partition $\sigma\in\mathsf{Aut}(\mathcal{D})$ iff perfect-matching $D=\big\{\xi=\{k_\xi,\ell_\xi\}_{k\neq\ell};|k_\xi\cap D|=1\big\}$; $\mathcal{D}=\{D,\forall\,k_\xi\}$, $\overline{\mathcal{M}}_g$ orientable compact, X closed, $g\gg$.



That is, for all $\sigma^{a_{k_\xi}b_{k_\xi}}:=1$ if $k_\xi\cap D$, resp. 0 else, then σ implies

$$\frac{1}{2}\,|\partial D=\mathcal{V}_X|\;=\;\frac{|\mathsf{Aut}(\mathcal{D})|\cdot|\mathcal{D}|^{-1}\cdot\prod_{\xi=1}^{n}|\sigma_{2\xi-1}\sigma_{2\xi}|}{\exp\big(n\ln 2+\sum_{m=1}^{n-1}\ln m\big)}\;=\sum_{k_\xi}\sigma^{a_{k_\xi}b_{k_\xi}}$$

$|\mathcal{V}_X=a_{k_\xi}\cap D|=2n\in\mathbb{N}$; $\sigma=(\sigma_1,\ldots,\sigma_{2n})\,|\,\widetilde{\sigma}=\sigma:\sigma_{k_{2\xi}}=\sigma_{2\xi}>\sigma_{k_{2\xi-1}}=\sigma_{2\xi-1}$ where $X\subset\overline{\mathcal{M}}_g$ is CW cell-complex i.e. face $\approx$ topological disk i.e. no hole.

Derivation. (i) $|\{\widetilde{\sigma}\}|^{1/|\mathscr{D}|} \leqslant \left((2n)!\cdot 2^{-((1/\varepsilon)\,\mathsf{mod}\,c(X))}\cdot e^{\ln(a(X)\cdot b(X))}\right)^{1/(2|\mathscr{D}|)}$

where
$$\min(\deg(X)) \geqslant \frac{n!\cdot a(X)\cdot b(X)}{\lfloor 2n-3\rfloor !! \,=\, \prod_{m=0}^{\lfloor n\rfloor -2} 2m+1} \;\Bigg|\; \begin{matrix} a,b,c\in\mathbb{R}^+ \\ n\geqslant 2 \end{matrix}$$

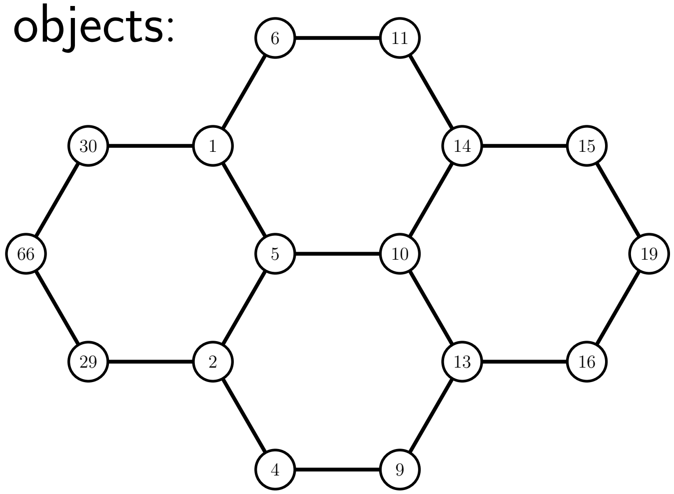
$\mathcal{S}_n \cong \{(\sigma_1,\ldots,\sigma_{2n}),\ldots,(\sigma_{2n-1},\sigma_{2n},\ldots,\sigma_1,\sigma_2)\}$; $\sigma \in \mathcal{S}_{2n} \longrightarrow \mathsf{Im}(\mathsf{Aut}(\mathscr{D}))$

$\mathcal{S}_2^n \cong \{(\sigma_1,\ldots,\sigma_{2n}),\ldots,(\sigma_2,\sigma_1,\ldots,\sigma_{2n},\sigma_{2n-1})\}$; $[\sigma] = \mathsf{Aut}(\mathscr{D})/(\mathcal{S}_n\times\mathcal{S}_2^n)$

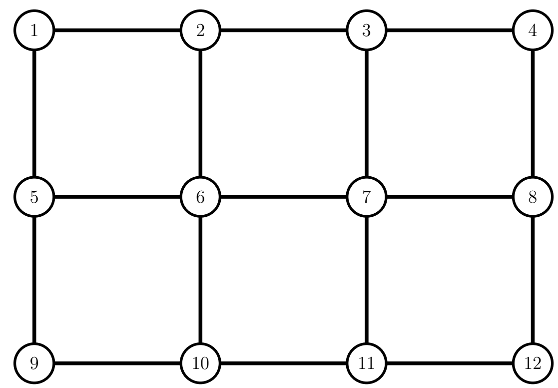
(ii) $|\{[\sigma]\}| = |\mathcal{S}_n\times\mathcal{S}_2^n|^{\,|\mathsf{Aut}(\mathscr{D})/(\mathcal{S}_n\times\mathcal{S}_2^n) = [\sigma]\cong\{\widetilde{\sigma}\}| = |\mathscr{D}|/\prod_{\xi=1}^{n}|\sigma_{2\xi-1}\sigma_{2\xi}|}$

(iii) $\binom{2n}{2} = 2\left(\frac{(2n)!}{n!2^n}\right) = \frac{2^{n+1}}{\sqrt{\pi}}\Gamma\left(n+\frac{1}{2}\right) = |\mathscr{E}_X|$ for complete graph $X = K_{2n}$

on objects:



- (regular) hexagonal grid domains.



- square grid domains.

By $\mathbb{E}[\sigma^{a_{k_\xi} b_{k_\xi}} \sigma^{\alpha_{\ell_\eta} \beta_{\ell_\eta}}] = \mathbb{E}[\sigma^{\alpha_{k_\xi} \beta_{k_\xi}}]$ iff $\xi = \eta$, $k = l$, $\{a, b\} = \{\alpha, \beta\}$, resp. zero if $\bigcap_{\xi \neq \eta, k \neq \ell} (a_{k_\xi}, b_{\ell_\eta}, D) \neq \emptyset$ or D dimers share vertex: Dimer-dimer correlation (conditional probability), for (Boltzmann) weights ω, is local observable

$$\left\langle \prod_{\xi=1}^{m} \sigma^{a_{k_\xi} b_{k_\xi}} \right\rangle \stackrel{\text{def}}{=\!=} \text{Prob}(a_{k_1} \cap D_{k_1}, b_{k_1} \cap D_{k_1}, \ldots, a_{k_m} \cap D_{k_m}, b_{k_m} \cap D_{k_m})$$

which equals

$$\mathbb{E}\left[\prod_{\xi=1}^{m} \sigma^{a_{k_\xi} b_{k_\xi}}\right] = \sum_{D_{k_\xi}} \prod_{\xi=1}^{m} \sigma^{a_{k_\xi} b_{k_\xi}} \times \text{Prob}(D_{k_\xi}) = \frac{\sum\limits_{D_{k_\xi}} \prod\limits_{\xi=1}^{m} \sigma^{a_{k_\xi} b_{k_\xi}} \prod\limits_{k_\xi \cap D_{k_\xi}} \omega_{k_\xi}}{\sum\limits_{D} \prod\limits_{k_\xi \cap D} \omega_{k_\xi}} = \frac{1}{\mathcal{Z}} \times \mathcal{Z}_{\widetilde{k_m}}$$

$$\neq 0 \iff \frac{1}{\mathcal{Z}} \sum_{D_{k_\xi} \cap (a_{k_1} \neq a_{k_2}, \ldots a_{k_m}, b_{k_1}, \ldots, b_{k_m})} \omega_{D_{k_\xi}} \qquad \Bigg| \quad \omega_D = \prod_{k_\xi \cap D} \omega_{k_\xi} = \prod_{k_\xi \cap D} e^{-\frac{\Xi_{k_\xi}}{\mathcal{K} T}} = e^{-\frac{\Xi_D}{\mathcal{K} T}}$$

$$= \text{Prob}(D) \iff D = \bigcup_{D_{k_\xi}} \bigcap_{k_\xi} \{a_{k_\xi}, D_{k_\xi}\} \qquad \Bigg| \quad \Xi_D = \sum_{k_\xi \cap D} \Xi_{k_\xi}, \quad \mathcal{Z} \stackrel{\text{def}}{=\!=} \sum_{D} \prod_{k_\xi \cap D} \omega_{k_\xi}$$

for strict-sense positive partition function $\mathcal{Z}$ on dimer energy

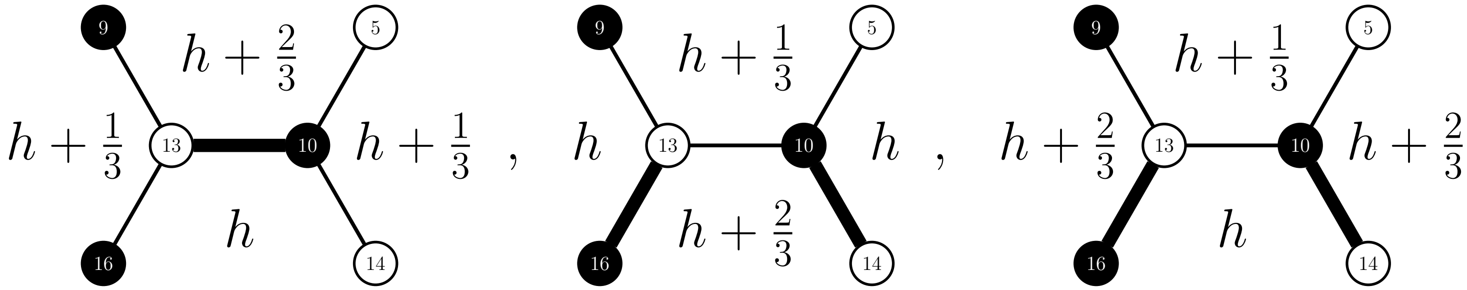
$$\Xi : \mathscr{E}_X \longrightarrow \mathbb{R}^+ \mid (a_{k_\xi} \cap D, b_{k_\xi} \cap D) \longmapsto \Xi_{k_\xi}.$$

Definition. The space $\mathcal{H}_X$ of height function h_D or h is the whole of $\mathbb{Z}$:

$$\boldsymbol{\mathcal{H}_X \overset{\text{def}}{=\!=} \{h_D\colon \mathscr{F}_X \longrightarrow \mathbb{Z}\}} \;\; | \;\; \boldsymbol{\mathscr{D}} \longleftarrow\!\!\shortmid \;\; \textbf{\textit{Bipartite surfaces}}$$

$$h(\mathscr{F}_i) = \begin{cases} h(\mathscr{F}_{i-1}) + 1/3 & \text{if } \; i^{\bullet}_{\xi} \text{ is left on crossing } \xi \not\cap D \\ h(\mathscr{F}_{i-1}) - 1/3 & \text{if } \; i^{\circ}_{\xi} \text{ is left on crossing } \xi \not\cap D \end{cases}; \quad h(\mathscr{F}_0) = 0.$$

Derivation. $\mathcal{H}_X$ is given on the bipartite hexagonal $X \subset \mathbb{R}^2$ by:



for any perfect-matching $D \in \mathscr{D}$, and base-face normalization $h_D(\mathscr{F}_0) = 0$.

Theorem. For all $\mathcal{H}_X$: (i) $h_D = h$ i.e. h_X is independent of D and path T^*.

(ii) Curl sum $d_X = \sum_{\mathscr{F}} d_{\mathscr{F}} = \sum_{\mathscr{F}} \sum_{\xi\colon \xi \cap \partial\mathscr{F}} \boldsymbol{\omega}_{i_\xi j_\xi} = 0$ iff $\mathscr{F}_X$ is all co-cycles.

Proof. Follows by divergence-free notion on X, that is,

$$d^*_{i_{D_1 D_2}} = d^*_{i_{D_1}} - d^*_{i_{D_2}} = 0 \text{ iff } \mathscr{F}_X \text{ is all co-cycles;}$$

$$d^*_{i_D} = d^*_i = \sum_j \sum_\xi \omega_{i_\xi j_\xi} \quad \Big| \text{ flow } \omega_{i_\xi j_\xi} = -\omega_{j_\xi i_\xi} = \begin{cases} +1 & \text{if } \; i\colon\, i^\bullet_\xi \cap D \\ -1 & \text{if } \; i\colon\, i^\circ_\xi \cap D \\ 0 & \text{else.} \end{cases}$$

Definition. Skew plane partition is sequence $\{\lambda \mid \lambda \supset \mu\}$ of diagonal slices

$$\lambda(t) = \big(\pi_{i,\,i+t} \in \mathbb{N} \;\mid\; i \geqslant \max(0,-t),\, \forall\, t \in \mathbb{Z}\big)$$

for finite monotone $(\pi_{ij} \geqslant \pi_{i+r,\,j+s}; r, s \geqslant 0)$ array in the generalized 3D partition array $\pi = (\pi_{ij} : (i,j) \in \mathbb{N}^2 \,|\, \pi_{ij} = 0, \forall\, i+j \gg 0)$ of X^* cubes π_{ij}.

Remark. π is uniquely determined by X^* bijection (projection) map

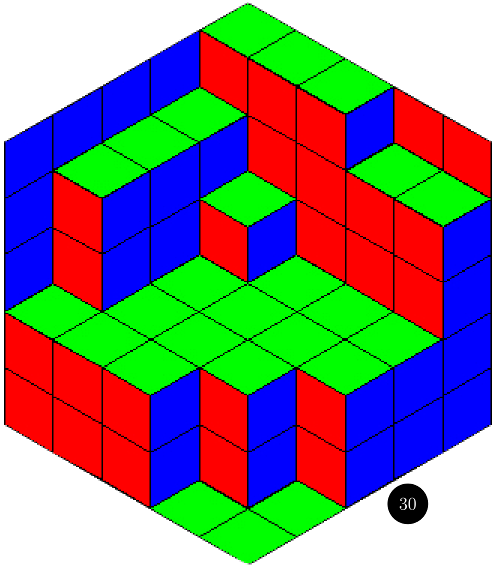
$$\mathbb{R}^3 \longmapsto \mathbb{R}^2 \supset \{(t,h)\}\colon\; t = y - x,\; h = z - (y+x)/2,\; \forall\,(x,y,z) \in \mathbb{R}^3$$

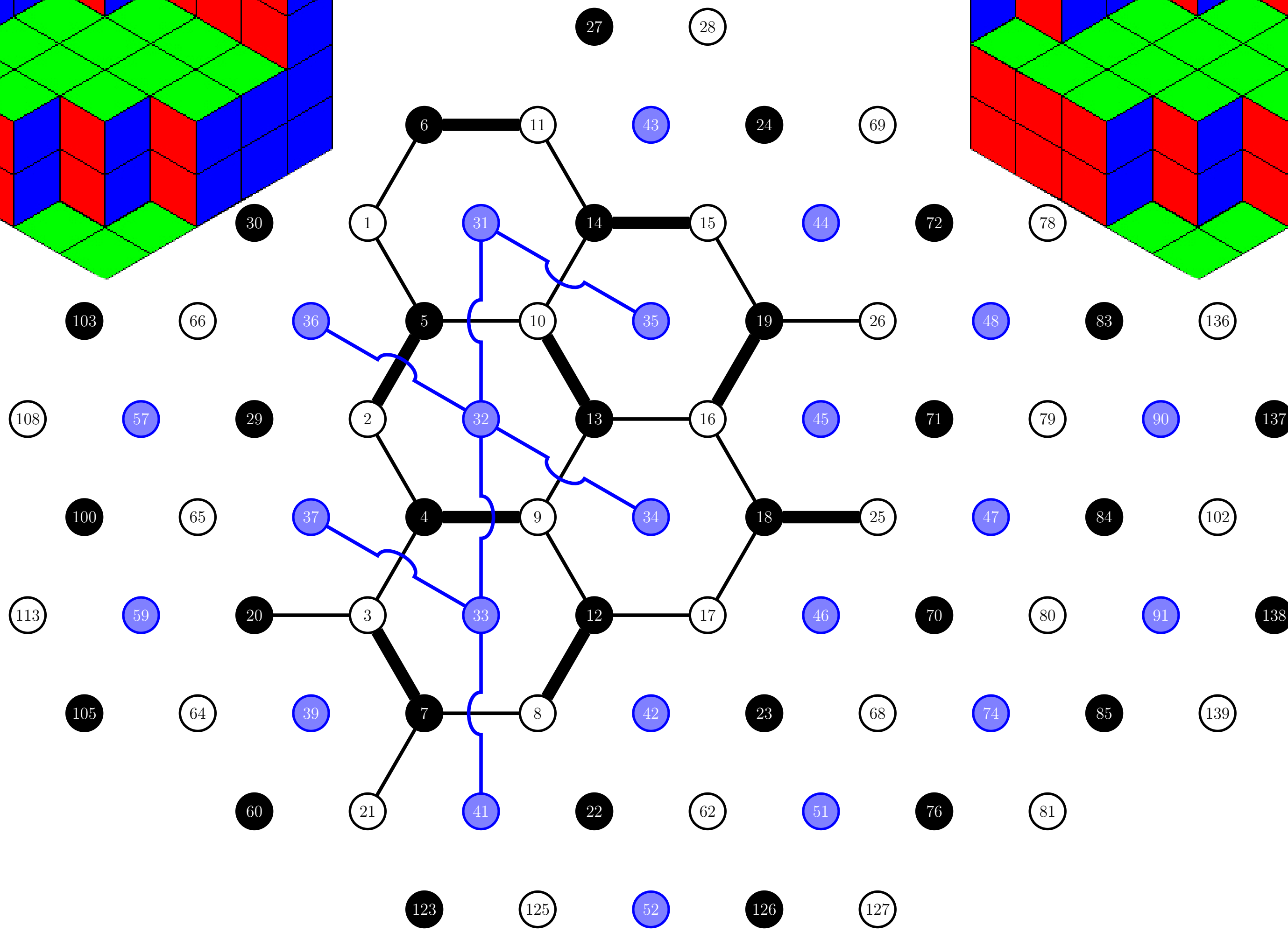
for all cubes mod $\mathbb{Z}^3_{\geqslant 0}$ projection, with boundary (base) condition $(0,0,0)$.

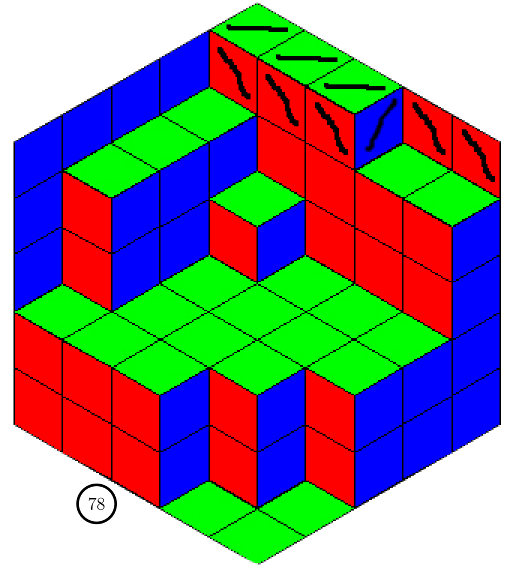
The centers of the horizontal hexagonal tiling is given by.

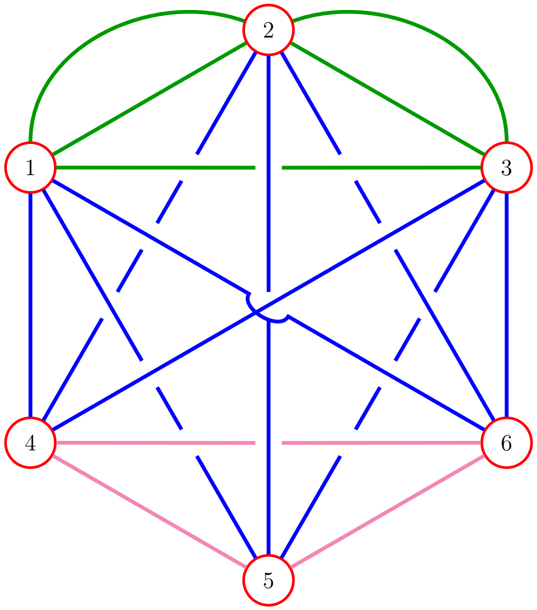
$$\pi_C = \big\{\big(i - j,\;\; \pi_{ij} - (i+j-1)/2\big)\big\} \,\subset\, \mathbb{Z} \times \frac{1}{2}\mathbb{Z}.$$

Cubes: 2D mixing algorithm









$$\left(\begin{array}{cccccc|cccc}
0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\hline
1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{array}\right)$$

$\mathscr{D}$ (left)

$\mathcal{H}_X$ (right)

Proposition (combinatorial correspondence).

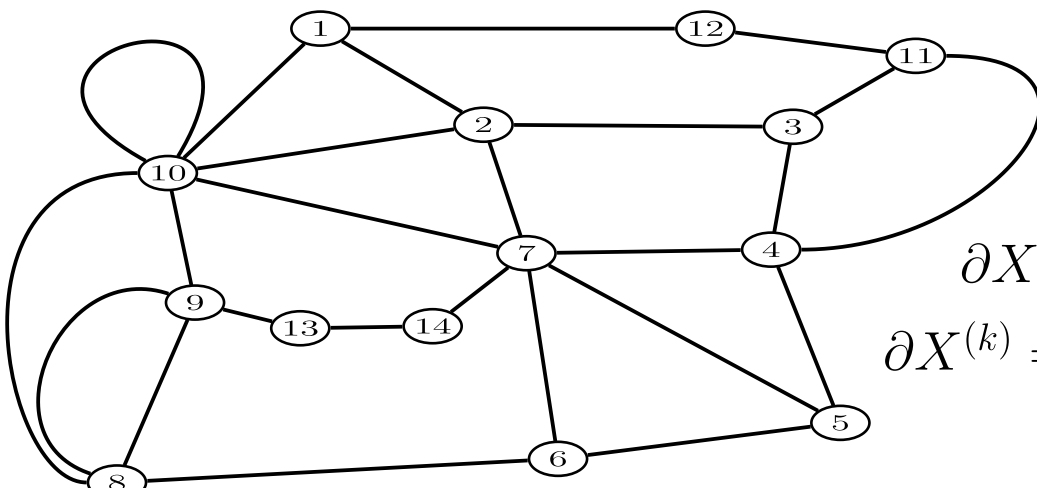
$$\{\textit{Dimers on } X\} \underset{\textit{bijection}}{\cong} \{\textit{height functions}\}.$$

Proof. For all $\mathfrak{D} \longleftarrow$ ***Discrete surfaces***, with spanning dual trees T^*,

family (Dimers) $\longleftrightarrow$ ***family (Tilings)***.

In particular, if $X \subset \mathbb{R}^2 =$ planar (no intersected edge) orientable, then

(i) 2D cell complex $X^{\mathbb{R}^2} = X \subset \mathbb{R}^2$:
0-cells, 1-cells, 2-cells $=$ vertices, edges, faces, resp.



Disjoint interiors.

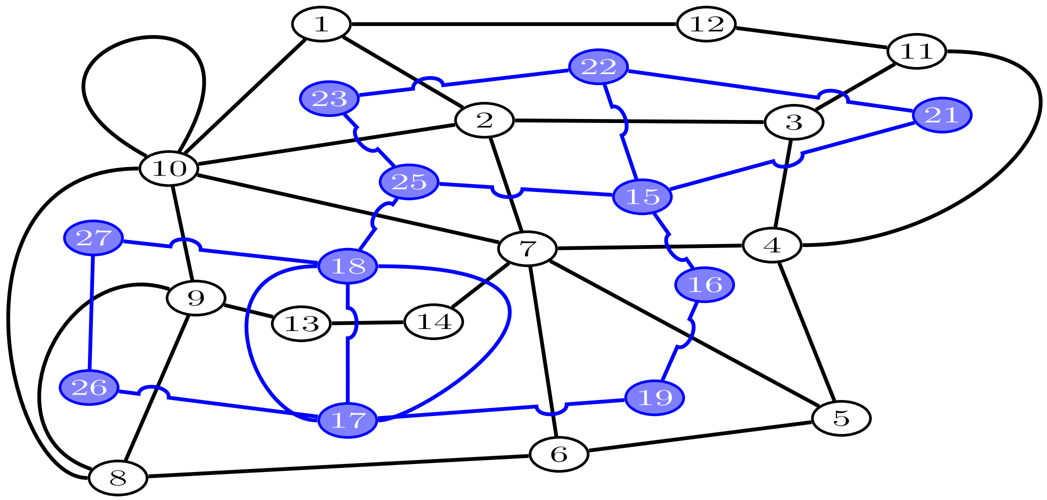
$\partial X^{(k)} = ((k-1) \bmod 2)$-cell

$\partial X^{(k)} =$ *boundary of two k-cells*, $k = 0, 1, 2.$

Remark. $X^{\overline{\mathcal{M}}_g}$: 1-skeleton CW complex (orientable compact decomposition).

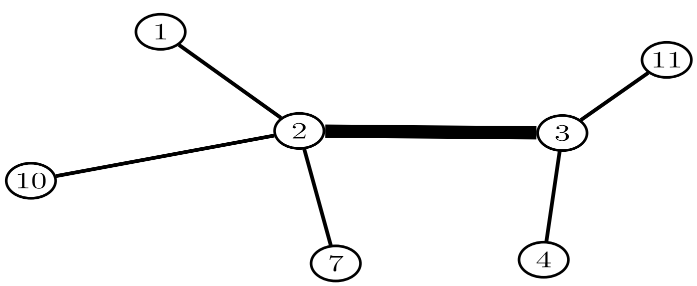
(ii) 2D dual cell complex X^*:

0-cells, 1-cells, 2-cells $=$ resp. "*centers*" of 2-cells, 1-cells, 0-cells of X.

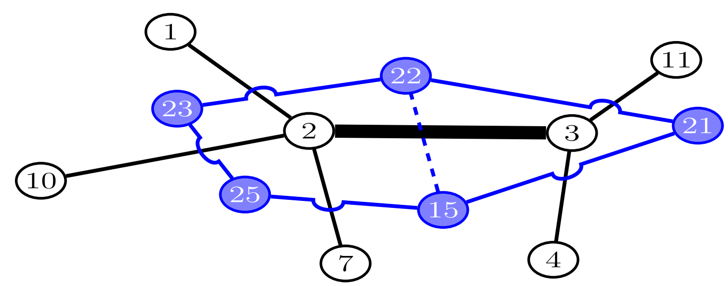


$X^* =$ dual cell complex to X.

(iii) For a dimer on X:



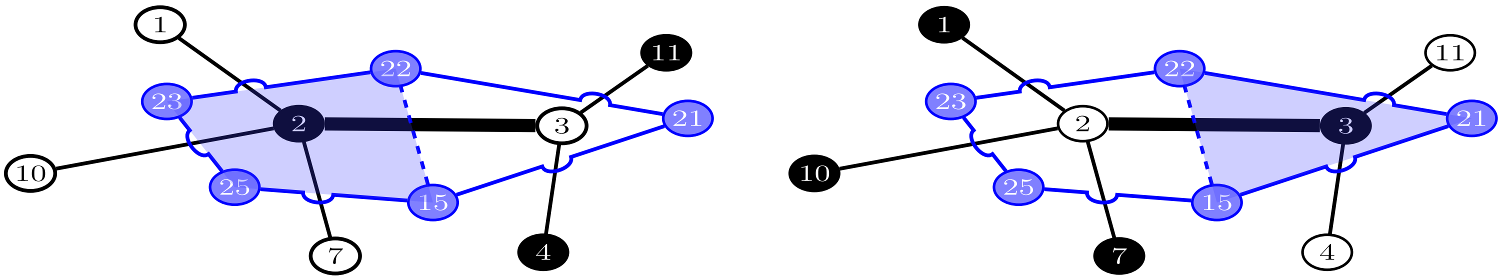
Unique pair of 2-cells on X^* share:



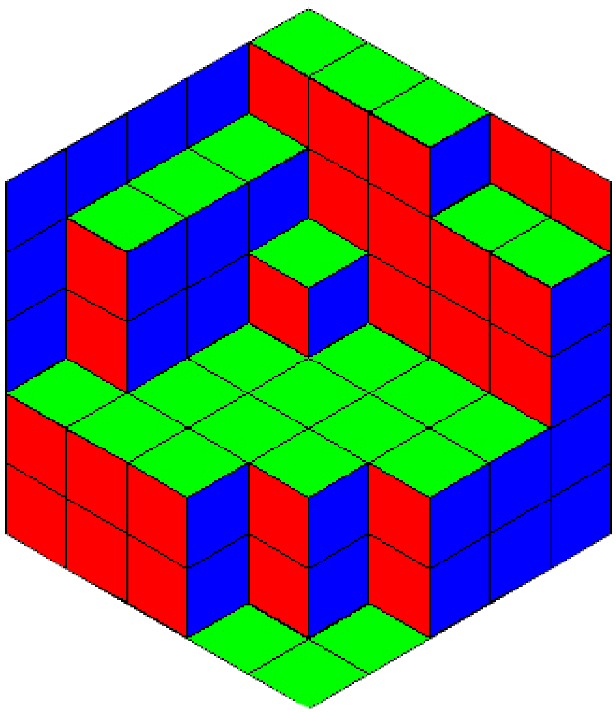
(iv) Therefore, the global bijection:

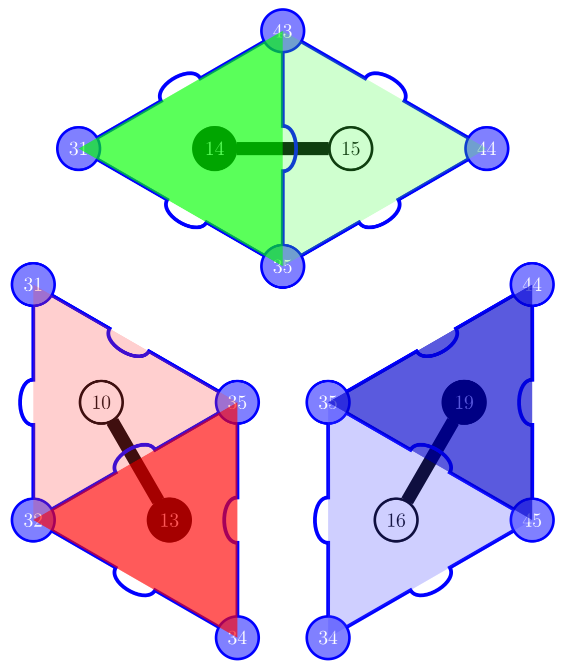
$$(\text{Dimers on } X) \longleftrightarrow \left(\begin{array}{l}\text{Tilings of } X^* \text{ by} \\ \textit{unique pair} \text{ of } 2\text{-cells}\end{array}\right). \qquad \square$$

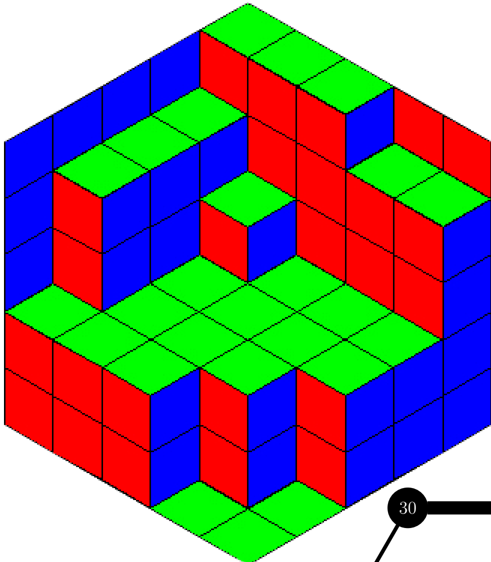
Remark. On bipartite graph, two-color tiles are admissible:



(Below: one-color tiles to the left, and two-color tiles to the right)

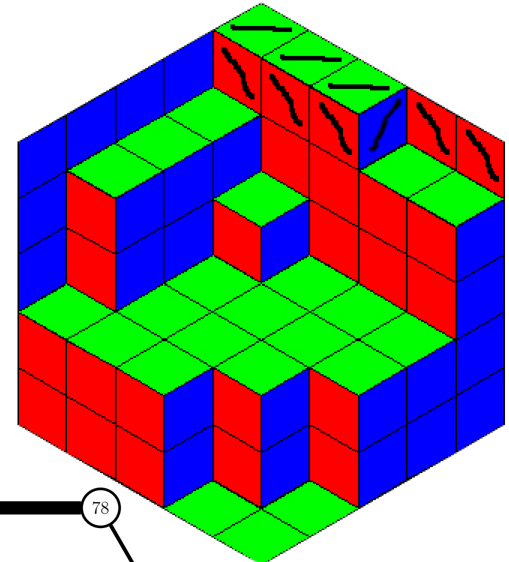


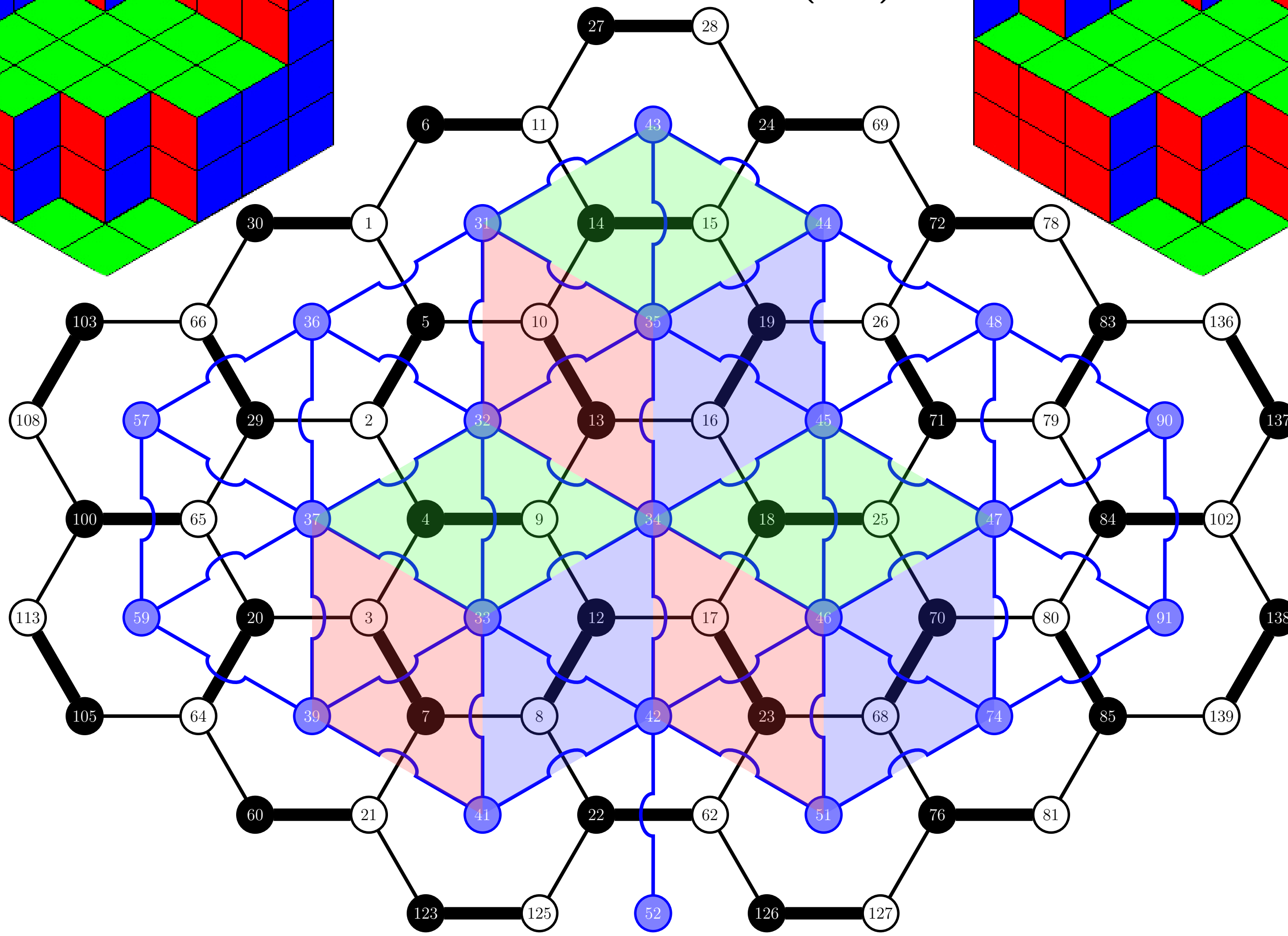




Cubes: 2D rhombus tiling

3D projection $\pi = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$





Lemma. $\mathsf{Prob}(D) = \dfrac{1}{\mathcal{Z}} \prod_{\mathcal{F}} q_{\mathcal{F}}^{h_D(\mathcal{F})} \;\Big|\; \mathcal{Z} = \sum_{D} \prod_{\mathcal{F}} \Big(\prod_{\xi \cap \partial\mathcal{F}} \omega_{\xi}^{\varepsilon_{\xi}^{\mathcal{K}}(\mathcal{F})} \Big)^{h_D(\mathcal{F})}$

where $q_{\mathcal{F}} =$ "essential" invariant parameter; $\varepsilon_{\xi}^{\mathcal{K}}$, $\varepsilon_{\partial\mathcal{F}} = \varepsilon_{\partial X}$, $\varepsilon^{-}_{\varepsilon_{\partial\mathcal{F}}}$ are edge, fixed (counterclockwise) boundary, counter orientation, $\forall\, \mathcal{F} \in \mathcal{F}_X \mid i_{\xi} \neq j_{\xi}$; $\varepsilon_{\xi}^{\mathcal{K}}(\mathcal{F}) = -1 \equiv \xi^{\swarrow}(\mathcal{F})$ (or $+1 \equiv \xi^{\nearrow}(\mathcal{F})$) if $\varepsilon_{\xi}^{\mathcal{K}} \in \varepsilon^{-}_{\varepsilon_{\partial\mathcal{F}}}$ (resp. $\varepsilon_{\xi}^{\mathcal{K}} \in \varepsilon_{\partial\mathcal{F}}$).

Proof. $\mathsf{Prob}(D)$ is well-defined. □

Theorem.
$D \sim h \colon \mathcal{F}_X \longrightarrow \mathbb{Z} \;\Big|\; \mathsf{Prob}(h) = \dfrac{1}{\mathcal{Z}} \prod_{\mathcal{F}} q_{\mathcal{F}}^{h(\mathcal{F})},\; \mathcal{Z} = \sum_{h} \prod_{\mathcal{F}} \Big(\prod_{\xi \cap \partial\mathcal{F}} \omega_{\xi}^{\varepsilon_{\xi}^{\mathcal{K}}(\mathcal{F})} \Big)^{h(\mathcal{F})}.$

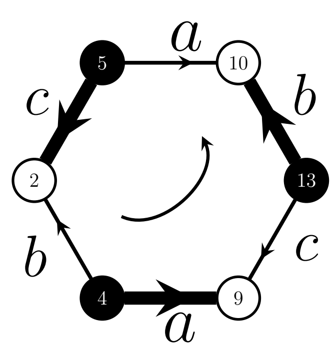
Proof. Follows by prior proposition and lemma. □

Remark. $\mathsf{Prob}(D) =$ "gauge" invariant measure: $\omega_{\xi} \longmapsto s(\xi_{+})\, \omega_{\xi}\, s(\xi_{-})$.

Cases.

(i) Uniform distribution:

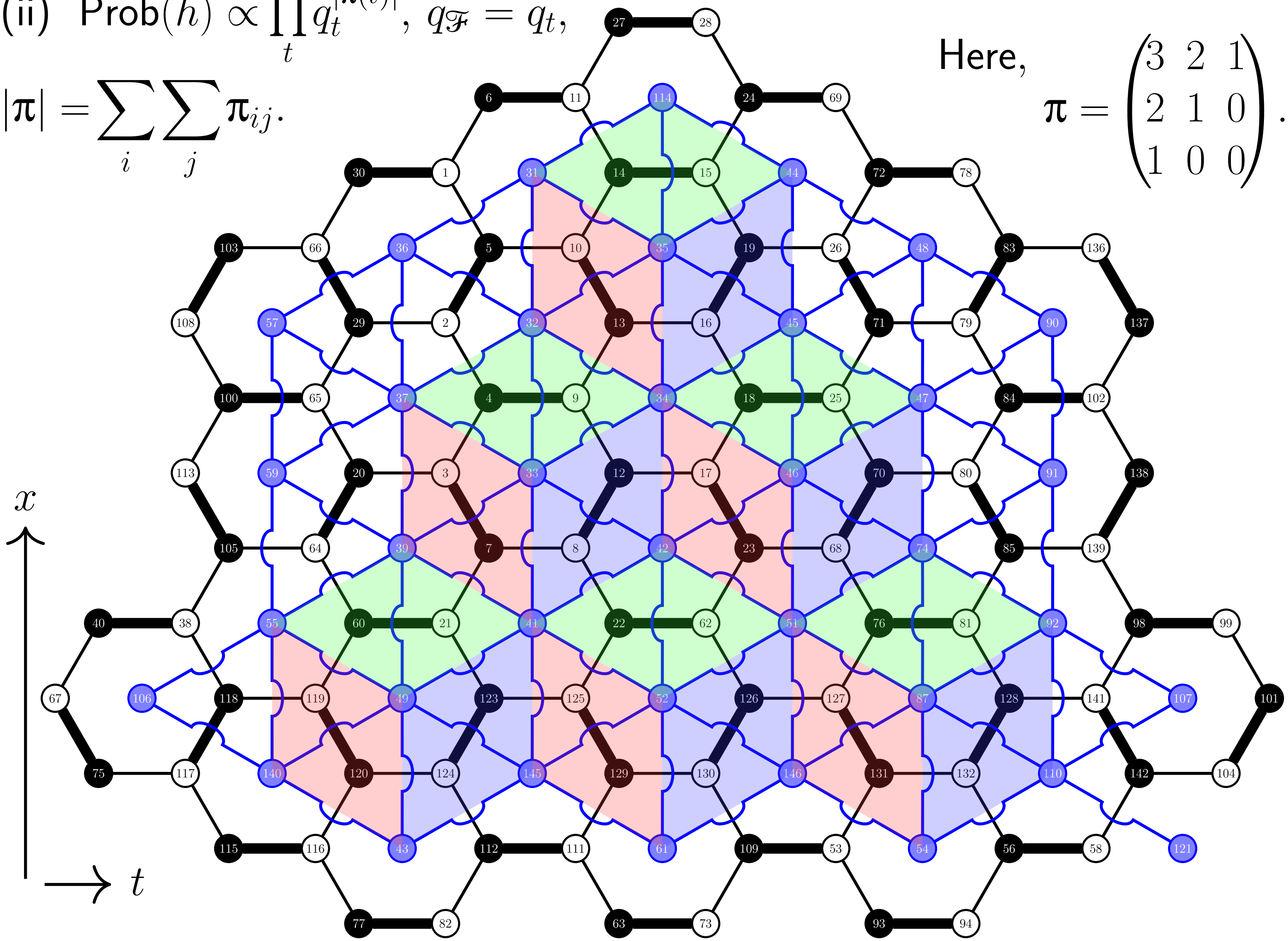
$$q_{\mathcal{F}} = 1 = a^{-1} b\, c^{-1} a\, b^{-1} c.$$



(ii) $\mathrm{Prob}(h) \propto \prod_t q_t^{|\pi(t)|}$, $q_{\mathcal{F}} = q_t$,

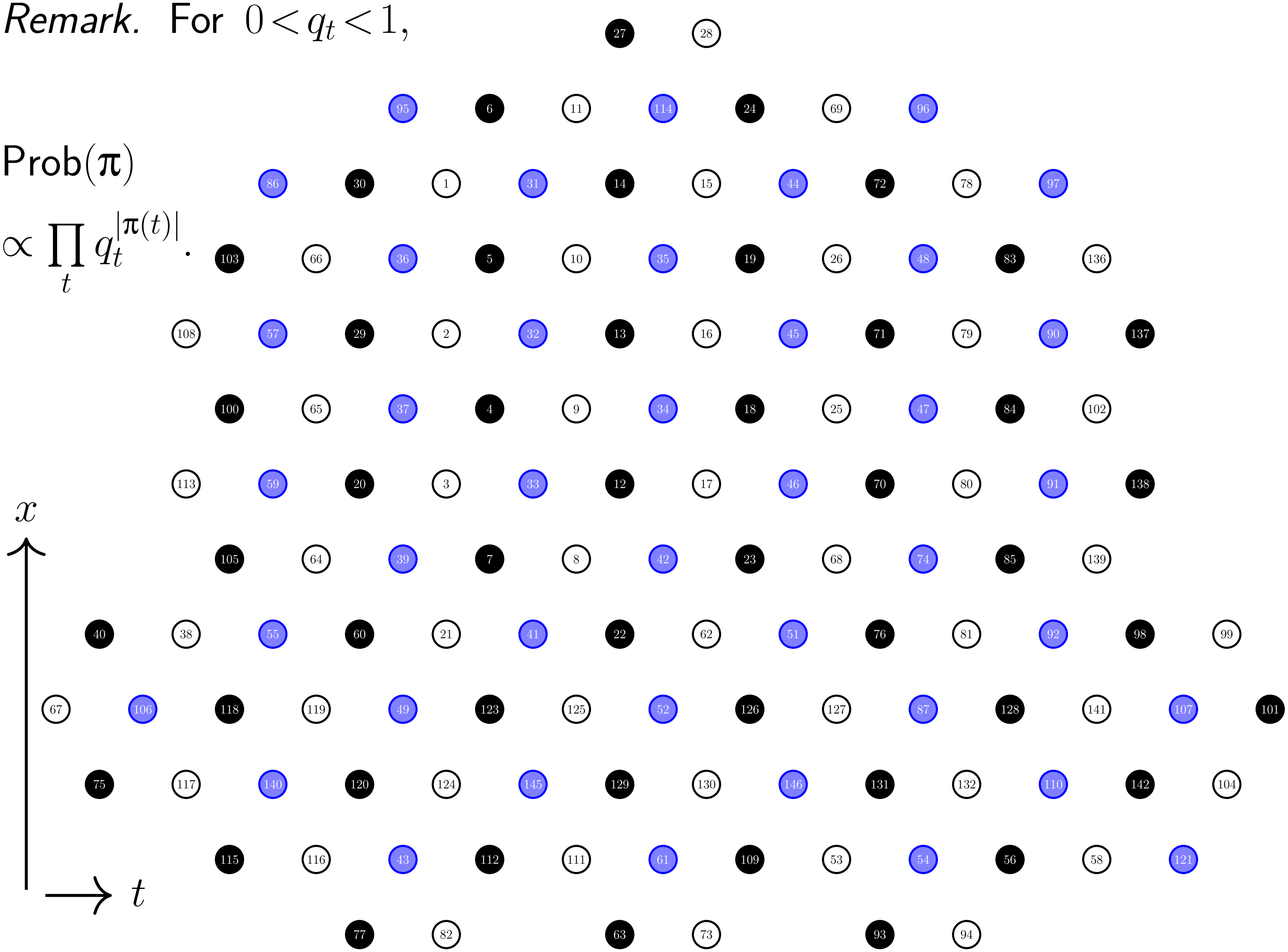
$|\pi| = \sum_i \sum_j \pi_{ij}$.

Here, $\pi = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.



Remark. For $0 < q_t < 1$,

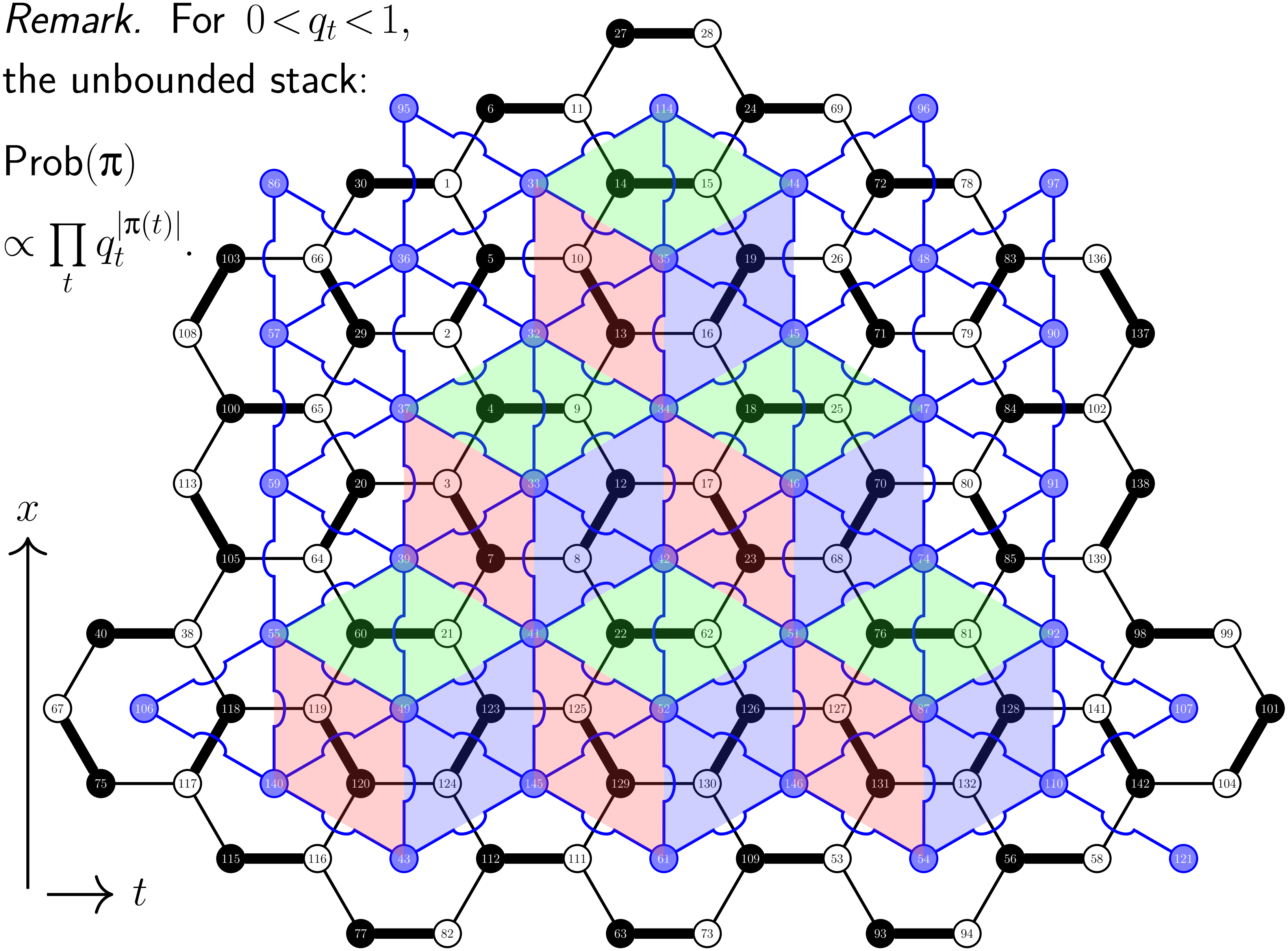
$$\text{Prob}(\pi) \propto \prod_t q_t^{|\pi(t)|}.$$



Remark. For $0 < q_t < 1$, the unbounded stack:

$\mathrm{Prob}(\pi)$

$\propto \prod_t q_t^{|\pi(t)|}$.



1.2 What is known

1.2.1 Order of + and − Pfaffians in $\mathcal{Z}$ for fixed $g \geqslant 0$

Kasteleyn (1963). For $g=0$, $\mathcal{Z} = \pm$ Pfaffian of Kasteleyn matrix.

Kasteleyn (1963). For $g=1$, $\mathcal{Z} =$ linear in 4 Pfaffians; 3 "$+$", 1 "$-$".

Kasteleyn (1963). For $g \geqslant 2$, $\mathcal{Z} =$ conjecture: 2^{2g} Pfaffians, appearing mysteriously i.e. proof was not given, at least not published.

1.2.2 Combinatorial representation of + and − in $\mathcal{Z}$

Gallucio & Loebl (1999). $\mathcal{Z} := \pm 1$; $\overline{\mathcal{M}}_g$ compact orientable.

Tesla (2000). $\mathcal{Z} := \sqrt{-1}$ and ± 1; $\overline{\mathcal{M}}_g$ non-orientable.

Cimazoni & R. (2004, 2005). $\mathcal{Z} := \pm 1$ by spin structure.

Cimasoni (2006). $\mathcal{Z} := \sqrt{-1}$ by pin-minus structure for double-cover; $\overline{\mathcal{M}}_g$ non-orientable; a Tesla (2000) topological model $\cong$ spin structure's ± 1.

1.2.3 Asymptotics of bipartite observable (Pfaffians)

R. et al. (2006). For height functions $h \in \mathbb{Z}$, face-weights $q_{\mathscr{F}}$, $\forall\, g \geqslant 2$,

$$\mathcal{Z}(\text{bipartite}) = \text{Const.} \times \sum_{h} \prod_{\mathscr{F}} q_{\mathscr{F}}^{h(\mathscr{F})} \quad \Big| \quad \mathcal{Z} = \frac{1}{2^g} \sum_{\mathfrak{T} \in S(\overline{\mathcal{M}}_g)} \text{Arf}(q_{\mathfrak{T}}^{\mathscr{K}}) \cdot \text{Pf}(X_{\mathfrak{T}}^{\mathscr{K}}).$$

And, as $|X| \longrightarrow \infty$, $q_{\mathscr{F}} \longrightarrow 1$, in Seiberg-Witten conjecture (Gaussian field theory) entropy, $\mathcal{Z}$ is scaling-limit path integral:

$$\mathcal{Z} = \int \exp\left\{ -\frac{1}{2} \left(\int_{\overline{\mathcal{M}}_g} (\partial\Phi)^2\, d^2x + \int_{\overline{\mathcal{M}}_g} \lambda(x)\, \Phi(x) \right) \right\}$$

where the term $q_{\mathscr{F}}^{h(\mathscr{F})}$ contributes to **R.H.S** linear multiple $\lambda(x)\,\Phi(x)$ by:

$$q_x = \xi^{-\varepsilon \cdot \lambda(x)} \quad \Big| \ \varepsilon = \text{lattice step};\ \lambda = \text{logarithmic scale, as } \varepsilon \longrightarrow 0.$$

Moreover, in Alvarez-Gaumé, Moore, Nelson & Vafa (1986), studying Fermi and Bose partition correspondence on Riemann surfaces,

$$\textbf{R.H.S.} \sim \sum_{\mathfrak{T} \in S(\overline{\mathcal{M}}_g)} \text{Arf}(\mathfrak{T}) \times |\Theta(z \,|\, \mathfrak{T})|^2 \quad \Big| \ \omega \text{ determines } z.$$

Remark. Conjecture (critical-weight): In large thermodynamic scaling limit asymptotics, the observable decaying linearly goes to

$$e^{\textsf{Volume}} \times \textbf{the free energy}$$

where the next leading term is sum of Θ functions, such that the square of each Θ function is next leading asymptotics of each of the Pfaffians.

The conjecture was confirmed by:

(i) **Ferdinand (1967).** *On square-grid torus.*

(ii) **Costa-Santos & McCoy (2002).** *Numerically*:

$$\mathsf{Arf}(\mathfrak{T}) \times |\Theta(z \,|\, \mathfrak{T})|^2 , \quad \forall\, g \geqslant 2 .$$

That is, the conjecture works, but no proof yet i.e. still a conjecture.

Remark. (i) $\mathscr{Z}$ is glueable (summable) on boundary spins of bounded surface.

(ii) "Higher" spin-structure is unknown, perhaps a para-polynomial theory.

(iii) Observable method is a non-deterministic sophistication, unlike $\mathrm{d}\log \boldsymbol{\omega}$.

Goal

1. Operators

(i) Prove $\mathcal{Z}$ invariant for all multiedge connected, genus g bipartite T^*

(ii) Prove the $\mathcal{O}(n^3)$ observable for all fixed sufficient-large genus $g \geqslant 0$

2. Vertex algebras

(i) Prove Grassmann kernel convergence for special genus g domain T^*

(ii) Obtain the $\mathbb{R}$ logarithmic scaling asymptotics by variational principle

(iii) State conjecture for the Green's function $\langle \cdot \rangle$ in large-deviation

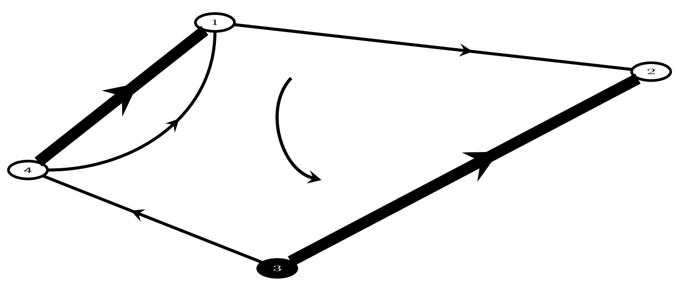
1.3 Orientation

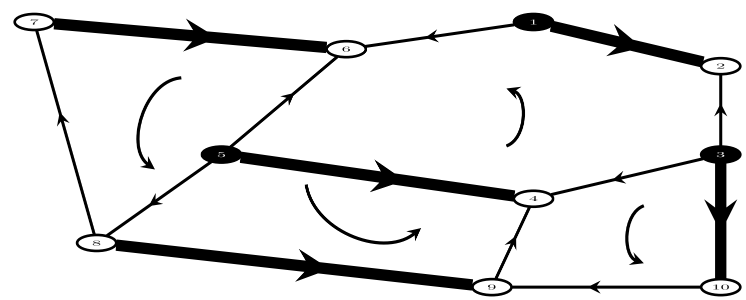
Definition. 1-skeleton CW complex (oriented compact cell-decomposition) $X \subset \overline{\mathcal{M}}_g$ is Kasteleyn $X^{\mathscr{K}}$ if $\forall\, \mathscr{F} \in \mathscr{F}_X$ orientation $\varepsilon^{\mathscr{K}}_{k_\xi} \,|\, i_{k_\xi} \neq j_{k_\xi}$ on a fixed (counterclockwise) boundary orientation $\varepsilon_{\partial\mathscr{F}} = \varepsilon_{\partial X}$, counter $\widetilde{\varepsilon}_{\partial\mathscr{F}} = \widetilde{\varepsilon}_{\partial X}$:

odd parity, $\rho^- = \mathbb{1}_{\mathscr{F}}\big(k^-_\xi\big) \pmod 2$

i.e. $\varepsilon^{\mathscr{K}}_{\mathscr{F}} = \prod\limits_{\xi \in \partial\mathscr{F}} \varepsilon^{\mathscr{K}}_{k_\xi}(\mathscr{F}) \;=\; -1$

$$\varepsilon^{\mathscr{K}}_{k_\xi}(\mathscr{F}) = \begin{cases} -1 \equiv k^-_\xi(\mathscr{F}) & \text{if } \varepsilon^{\mathscr{K}}_{k_\xi} \in \widetilde{\varepsilon}_{\partial\mathscr{F}} \\ +1 \equiv k^+_\xi(\mathscr{F}) & \text{if } \varepsilon^{\mathscr{K}}_{k_\xi} \in \varepsilon_{\partial\mathscr{F}}. \end{cases}$$





Given $X^{\mathscr{K}}$ for ω_{k_ξ} trivial otherwise, $\forall\, k$ edges connecting i_{k_ξ} and j_{k_ξ},

$$X^{\mathscr{K}}_{ij} = \sum_k \varepsilon^{\mathscr{K}}_{i_{k_\xi} j_{k_\xi}} \omega_{k_\xi} = -X^{\mathscr{K}}_{ji} \quad \Big| \quad X^{\mathscr{K}}_{ii} = 0; \;\; \varepsilon^{\mathscr{K}}_{i_{k_\xi} j_{k_\xi}} = \begin{cases} -\varepsilon^{\mathscr{K}}_{k_\xi} & \text{if } k_\xi \text{ is } j_{k_\xi} \text{ to } i_{k_\xi} \\ \varepsilon^{\mathscr{K}}_{k_\xi} & \text{if } k_\xi \text{ is } i_{k_\xi} \text{ to } j_{k_\xi}. \end{cases}$$

Remark. Bipartite Kasteleyn orientation is *well-defined* in hexagonal lattice.

Derivation. If $\varepsilon^{\mathscr{K}}_{i_\xi j_\xi} = \varepsilon^{\mathscr{K}}_{j_\xi i_\xi} = 1$, then $(X^{\mathscr{K}}_{ij})$ is called the adjacency matrix (resp. weighted adjacency matrix), $\forall\, \omega_\xi = 1$ (resp. $\omega_\xi > 0$).

Derivation. Let $X^{\mathscr{K}} \subset \overline{\mathcal{M}}_g$ be bipartite, then

$$X^{\mathscr{K}}_{ij} = -X^{\mathscr{K}}_{ji} = \begin{cases} \omega_\xi & \text{if} \quad i_\xi \bullet\!\!\Longrightarrow\!\!\circ\, j_\xi \quad \text{or} \quad i_\xi \bullet\!\!\longrightarrow\!\!\circ\, j_\xi \\ -\omega_\xi & \text{if} \quad i_\xi \circ\!\!\Longleftarrow\!\!\bullet\, j_\xi \quad \text{or} \quad i_\xi \circ\!\!\longleftarrow\!\!\bullet\, j_\xi \\ 0 & \text{if} \quad i = j \quad \text{or} \quad \xi \neq \eta \quad \forall\, i_\xi\,,\, j_\eta \end{cases}$$

or

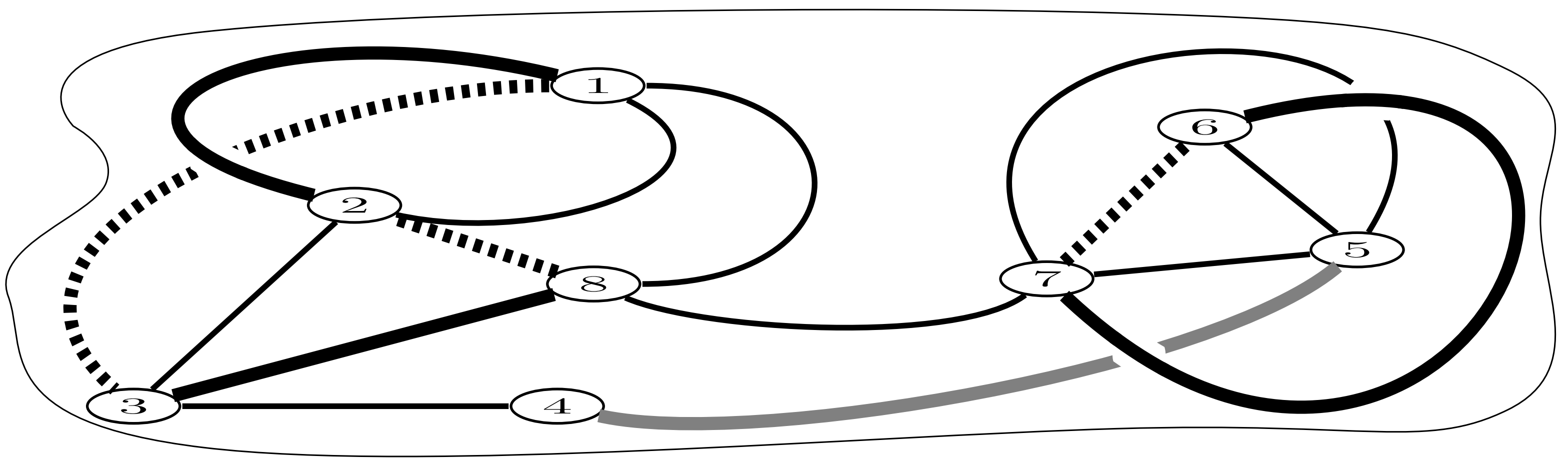
$$X^{\mathscr{K}}_{ij} = -X^{\mathscr{K}}_{ji} = \begin{cases} \omega_\xi & \text{if} \quad i_\xi \circ\!\!\Longrightarrow\!\!\bullet\, j_\xi \quad \text{or} \quad i_\xi \circ\!\!\longrightarrow\!\!\bullet\, j_\xi \\ -\omega_\xi & \text{if} \quad i_\xi \bullet\!\!\Longleftarrow\!\!\circ\, j_\xi \quad \text{or} \quad i_\xi \bullet\!\!\longleftarrow\!\!\circ\, j_\xi \\ 0 & \text{if} \quad i = j \quad \text{or} \quad \xi \neq \eta \quad \forall\, i_\xi\,,\, j_\eta. \end{cases}$$

The transition subgraph is symmetry $D_1 \Delta D_2 = D_1 \cup D_2 \setminus D_1 \cap D_2$ of 1-chain complex $\mathcal{C}^1(X^{\mathcal{K}}; \mathbb{Z}_2)$; 1-cycle homology $\mathcal{H}^1(X^{\mathcal{K}}; \mathbb{Z}_2) = \mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)$ class of all ordered, even-length $\eta = \sum_\alpha \sigma^{C_\alpha \cap D_1 \Delta D_2}$ simple closed transition paths

$$C_\alpha = \left(\sigma_{n_{\alpha-1}+1}, \ldots, \sigma_{n_\alpha}\right), \ \forall \alpha \in \mathbb{N}^+ \mid 1 \leqslant \alpha \leqslant \eta, \ n_0 = 0,$$

traversing $\sigma_{n_{\alpha-1}+1}, (\sigma_{n_{\alpha-1}+1}, \sigma_{n_{\alpha-1}+2}), \ldots, \sigma_{n_\alpha}, (\sigma_{n_\alpha}, \sigma_{n_{\alpha-1}+1})$ given by:

$$\left((\sigma_{n_{\alpha-1}+1}, \sigma_{n_{\alpha-1}+2}), \ldots, (\sigma_{n_\alpha-1}, \sigma_{n_\alpha})\right) \subseteq D_1$$
$$\left((\sigma_{n_{\alpha-1}+2}, \sigma_{n_{\alpha-1}+3}), \ldots, (\sigma_{n_\alpha}, \sigma_{n_{\alpha-1}+1})\right) \subseteq D_2 \,.$$



Remark. D_1, D_2 are equivalent if $|D_1 \Delta D_2| = 0 \in \mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$; $D_1, D_2 = 1$-chain in cell-complex $\mathcal{C}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)$; $\partial D_1, \partial D_2 = \mathcal{C}^0(\overline{\mathcal{M}}_g; \mathbb{Z}_2)$.

Lemma (sign). *The monomial sign for fixed sufficient large genus* g,

$$\varepsilon^{\mathscr{K}}_{D} = (-1)^{t(\sigma)} \prod_{\xi \in D} \varepsilon^{\mathscr{K}}_{\sigma_{2\xi-1}\sigma_{2\xi}} \;\Big|\; t(\sigma) := (\sigma_1, \ldots, \sigma_{2n}) \longrightarrow (1, \ldots, 2n)$$

is invariant of Aut$(\mathscr{D})$.

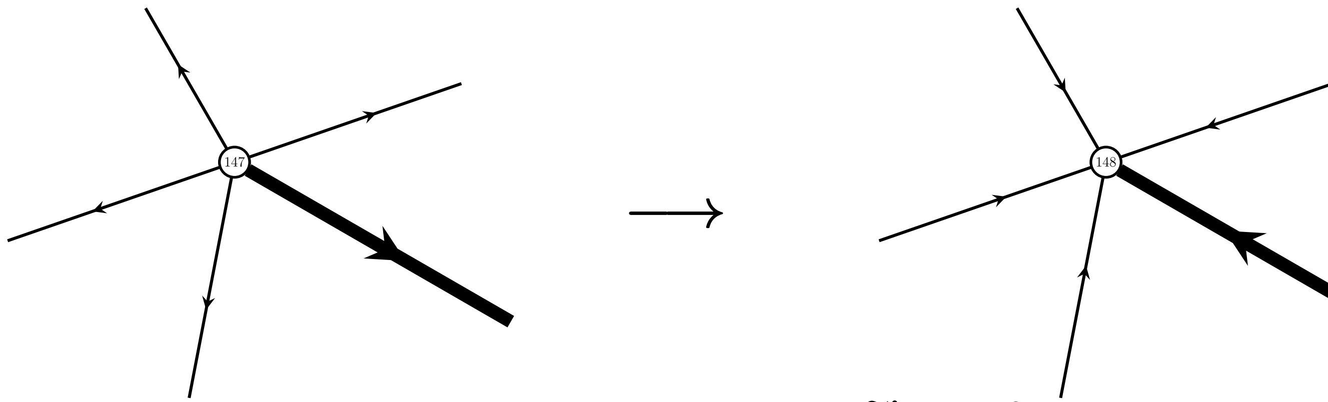
Proof. $\varepsilon^{\mathscr{K}}_{D}$ is Aut(D) invariant by transposition of $\sigma_{2\xi-1}\sigma_{2\xi}$, with $(-1)^{t(\sigma)}$. Now, let $D_1, D_2 \in \mathscr{D}$ orient from $\sigma_{2\xi-1}$ to $\sigma_{2\xi}$, resp. $\tau_{2\eta-1}$ to $\tau_{2\eta}$, in cyclic order of $\widetilde{\sigma}$, resp. $\widetilde{\tau}$, $\forall\, C_\alpha$ (transition even cycles). Then, exactly odd $\rho^- = \mathbb{1}_{C_\alpha}(\xi^-) = \mathbb{1}_{C_\alpha}(\xi^+)$, $\forall\, \alpha$. Hence, for all $\sigma_{2\nu-1}\sigma_{2\nu} = \tau_{2\nu-1}\tau_{2\nu}$ composition $\gamma = \sigma \circ \tau$,

$$+1 \;=\; \varepsilon^{\mathscr{K}}_{D_1}\,\varepsilon^{\mathscr{K}}_{D_2} = \prod_{\alpha}\;\prod_{\xi \in C_\alpha}\;\prod_{\eta \in C_\alpha} \varepsilon^{\mathscr{K}}_{\sigma_{2\xi-1}\sigma_{2\xi}}\,\varepsilon^{\mathscr{K}}_{\tau_{2\eta-1}\tau_{2\eta}} \prod_{\nu}\Big(\varepsilon^{\mathscr{K}}_{\sigma_{2\nu-1}\sigma_{2\nu}} = \varepsilon^{\mathscr{K}}_{\tau_{2\nu-1}\tau_{2\nu}}\Big)^2$$

$$= \prod_{\alpha}\;\prod_{\xi\,\vee\,\xi^* \in C_\alpha}\;\prod_{\eta\,\vee\,\eta^* \in C_\alpha} \varepsilon^{\mathscr{K}}_{\sigma_{2(\xi\,\vee\,\xi^*)\,-\,1}\;\sigma_{2(\xi\,\vee\,\xi^*)}}\;\varepsilon^{\mathscr{K}}_{\tau_{2(\eta\,\vee\,\eta^*)\,-\,1}\;\tau_{2(\eta\,\vee\,\eta^*)}}$$

$\Longrightarrow \varepsilon^{\mathscr{K}}_{D_1} = \varepsilon^{\mathscr{K}}_{D_2}$, for $\mathbb{1}_{C_\alpha}\big(\xi^{\swarrow} \vee \xi^{*\swarrow} \vee \eta^{\swarrow} \vee \eta^{*\swarrow}\big) = 1 \,(\mathrm{mod}\ 2)$, $\forall\,\alpha$, by $\xi^* \vee \eta^*$ i.e. $\varepsilon^{\mathscr{K}}_{D_1} = \varepsilon^{\mathscr{K}}_{D_2}, \forall\,\rho^- = \mathbb{1}_{C_\alpha}\big((\cdot)^{\swarrow}\big) \equiv \mathbb{1}_{C_\alpha}\big((\cdot)^{\nearrow}\big) = \rho^+$ through Aut(D_1) invariance, resp. Aut(D_2) invariance, $\forall\, D_1, D_2 \in \mathscr{D}$. □

Definition. Two orientations are equivalent if there exists reversing-map:



Theorem. *All Kasteleyn orientations of* $X^{\mathscr{K}} \subset \mathbb{R}^2$ *are equivalent.*

Proof. Given two Kasteleyn orientations $\mathscr{K}_-$, $\mathscr{K}_+$ marked by $\mathscr{K}_-$ (resp. $\mathscr{K}_+$) on ith end (resp. jth end) of ξ, $\forall\,\mathscr{F}$, $\varepsilon_{\partial\mathscr{F}} = \varepsilon_{\partial X}$, then

$$\varepsilon_\xi^{\mathscr{K}_-} = \varepsilon_\xi^{\mathscr{K}_+} \cdot \sigma_\xi^{\mathscr{K}_-\mathscr{K}_+}, \quad \varepsilon_\xi^{\mathscr{K}_+} = \varepsilon_\xi^{\mathscr{K}_-} \cdot \sigma_\xi^{\mathscr{K}_-\mathscr{K}_+} \quad \Big| \, \sigma_\xi^{\mathscr{K}_-\mathscr{K}_+} = \varepsilon_\xi^{\mathscr{K}_-} \cdot \varepsilon_\xi^{\mathscr{K}_+}$$

i.e. $\mathscr{K}_- \longrightarrow \mathscr{K}_+$ (resp. $\mathscr{K}_+ \longrightarrow \mathscr{K}_-$) by $\sigma_\xi^{\mathscr{K}_-\mathscr{K}_+}$ multiplying $\mathscr{K}_-$ (resp. $\mathscr{K}_+$) at every vertex; and, $\mathscr{K}_- \longleftrightarrow \mathscr{K}_+ \longleftrightarrow$ equivalence class $[\mathscr{K}]$ in simple reversal of orientations around vertices by $-1 = \sigma_\xi^{\mathscr{K}_-\mathscr{K}_+} := \pm 1$. $\square$

Corollary. *Equivalence class* $[\mathscr{K}]$ *is unique for* $X^{\mathscr{K}} \subset \mathbb{R}^2$.

Proof. $\exists$ one homotopy class of loops i.e. $\mathbb{R}^2$ trivial fundamental group. $\square$

Theorem. Kasteleyn orientation equivalence classes $[\mathscr{K}]$ are exactly 2^{2g}.

Proof. The isomorphisms $\{[\mathscr{K}]\}$ are in characteristic-2 field κ affine closure $\mathrm{Sym}^2_{\kappa}(V^{\wedge})$ of non-degenerate, skew-symmetric quadratic bilinear form

$$q(\alpha+\beta) = q(\alpha)+q(\beta)+\alpha\cdot\beta \;\Big|\; q\colon V\otimes V \longrightarrow \kappa,\ \forall\,\alpha,\beta\in\mathcal{H}^1 = V\otimes V$$

in first homology space $\mathcal{H}^1 \ni \alpha$, for

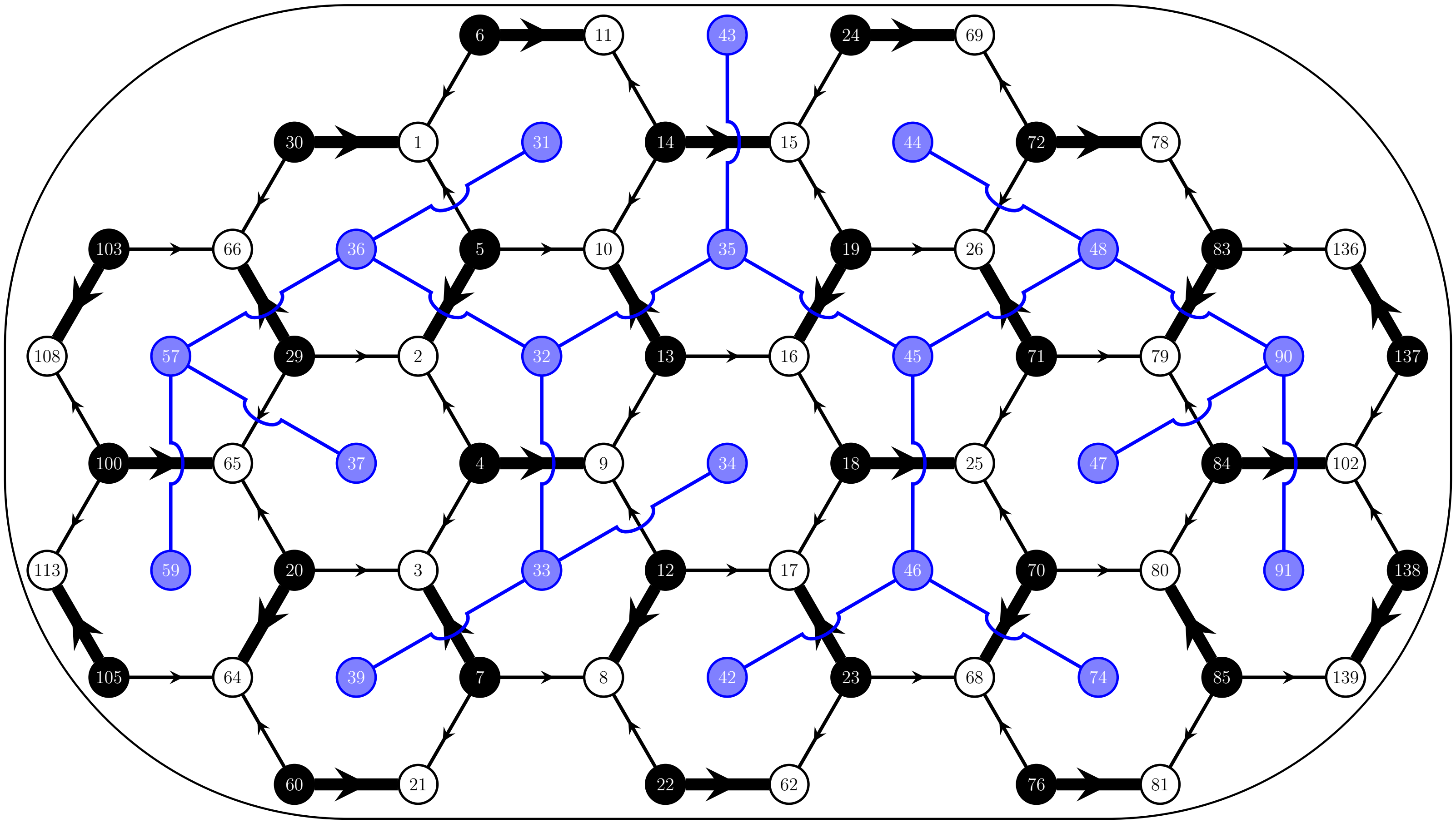
$$\frac{1}{\sqrt{|\mathcal{H}^1|}}\sum_{q\,\in\,(\mathcal{H}^1,\cdot)}(-1)^{\mathit{Arf}(q)+q(\alpha)} = 1 \;\Bigg|\; \mathit{Arf}(q) = \sum_{\{\xi,\eta\}} q(\xi)q(\eta)\ \in\ \kappa/f(\kappa)\subset\mathbb{Z}_2$$

where $\{\xi,\eta\}$ are symplectic basis pairs for symplectomorphisms $V\longrightarrow V$, Lang's isogeny $f\colon\kappa\longrightarrow\kappa \mid x\longmapsto x^2-x\in \mathsf{Gal}/\mathbb{F}_2$ (2-element Galois field).

By continuity $\psi\colon X^{\mathscr{K}}\longrightarrow\overline{\mathcal{M}}_g$, every $\overline{\mathcal{M}}_g\backslash\psi(X^{\mathscr{K}})$ connected-components (ψ-faces $\mathscr{F}$) $\approx$ open disk, i.e. $\chi(X^{\mathscr{K}}) = \chi(\overline{\mathcal{M}}_g)$ in Euler-Poincaré bound $|\mathscr{V}_{X^{\mathscr{K}}}|-|\mathscr{E}_{X^{\mathscr{K}}}|+|\mathscr{F}_{X^{\mathscr{K}}}| = \chi(X^{\mathscr{K}})\geqslant\chi(\overline{\mathcal{M}}_g)$. Vanishing composition $\partial_1\circ\partial_2$ of boundary operators $\partial_2\colon\mathcal{C}_2\longrightarrow\mathcal{C}_1$, $\partial_1\colon\mathcal{C}_1\longrightarrow\mathcal{C}_0$ for basis $\mathcal{C}_0,\mathcal{C}_1,\mathcal{C}_2$ of 2D cell-complex $\mathscr{V}_{X^{\mathscr{K}}}, \mathscr{E}_{X^{\mathscr{K}}}, \mathscr{F}_{X^{\mathscr{K}}}$, resp. implies 1-cycle space superset $\mathrm{Ker}(\partial_1)$ of 1-boundary space $\partial_2(\mathcal{C}_2)$. Hence, independent of $X^{\mathscr{K}}$ but depending only on genus g: $|\mathcal{H}^1(\overline{\mathcal{M}}_g;\mathbb{Z}_2)| = |\mathcal{H}^1(X^{\mathscr{K}};\mathbb{Z}_2)| = |\mathrm{Ker}(\partial_1)/\partial_2(\mathcal{C}_2)| = 2^{2g}$. $\square$

Theorem (existence). *Kasteleyn orientation exists* $\Longleftrightarrow \left|\mathscr{V}_{X\mathscr{K}}\right| =$ *even.*

Proof. Following a rooted spanning dual tree T^*:



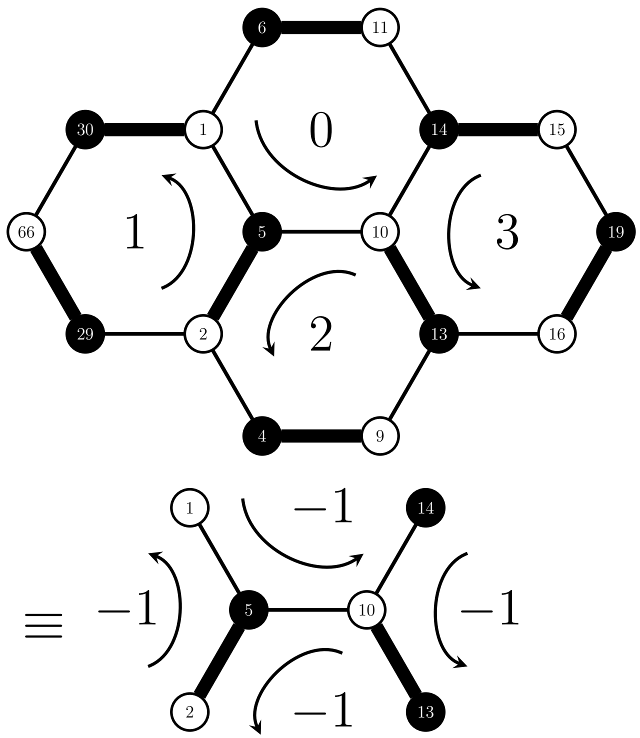
Reduce X to $\ll$ by $n \times n \longrightarrow \exp(\alpha n^2)$; and, arbitrarily orient every ξ not crossing T^*. Then, deleting ξ^* from leaves starting at root, make $\varepsilon_{\mathscr{F}}^{\mathscr{K}}$, $\forall \mathscr{F}$.

Now,

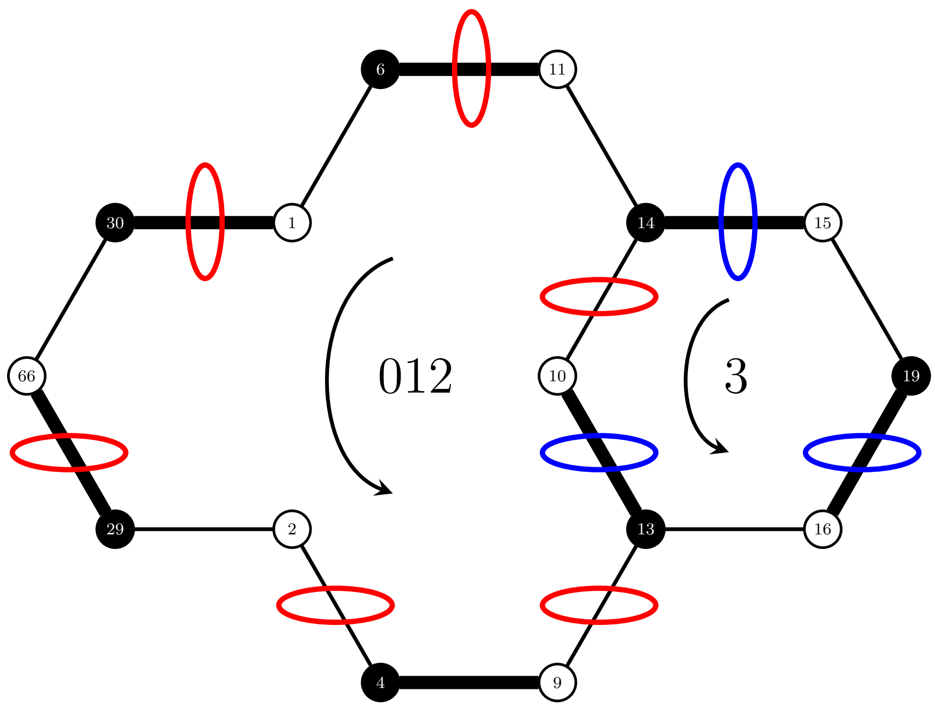
$$\prod_{\mathscr{F}\in\mathscr{F}_{X\mathscr{K}}}(-1)^{\left(\sum_{\swarrow}\xi^{\swarrow}(\mathscr{F})\right)} = (-1)^{\left(\sum_{\mathscr{F}\in\mathscr{F}_{X\mathscr{K}}}\sum_{\swarrow}\xi^{\swarrow}(\mathscr{F})\right)} = (-1)^{|\mathscr{E}_{X\mathscr{K}}|} \implies |\mathscr{V}_{X\mathscr{K}}| = \text{even}$$

by the Euler-Poincaré equality: $|\mathscr{V}_{X\mathscr{K}}| \overset{\text{mod } 2}{=\!=} |\mathscr{E}_{X\mathscr{K}} - \mathscr{F}_{X\mathscr{K}}|$. □

Remark. Deleted-vertex changes Kasteleyn to non-Kasteleyn at "hole":

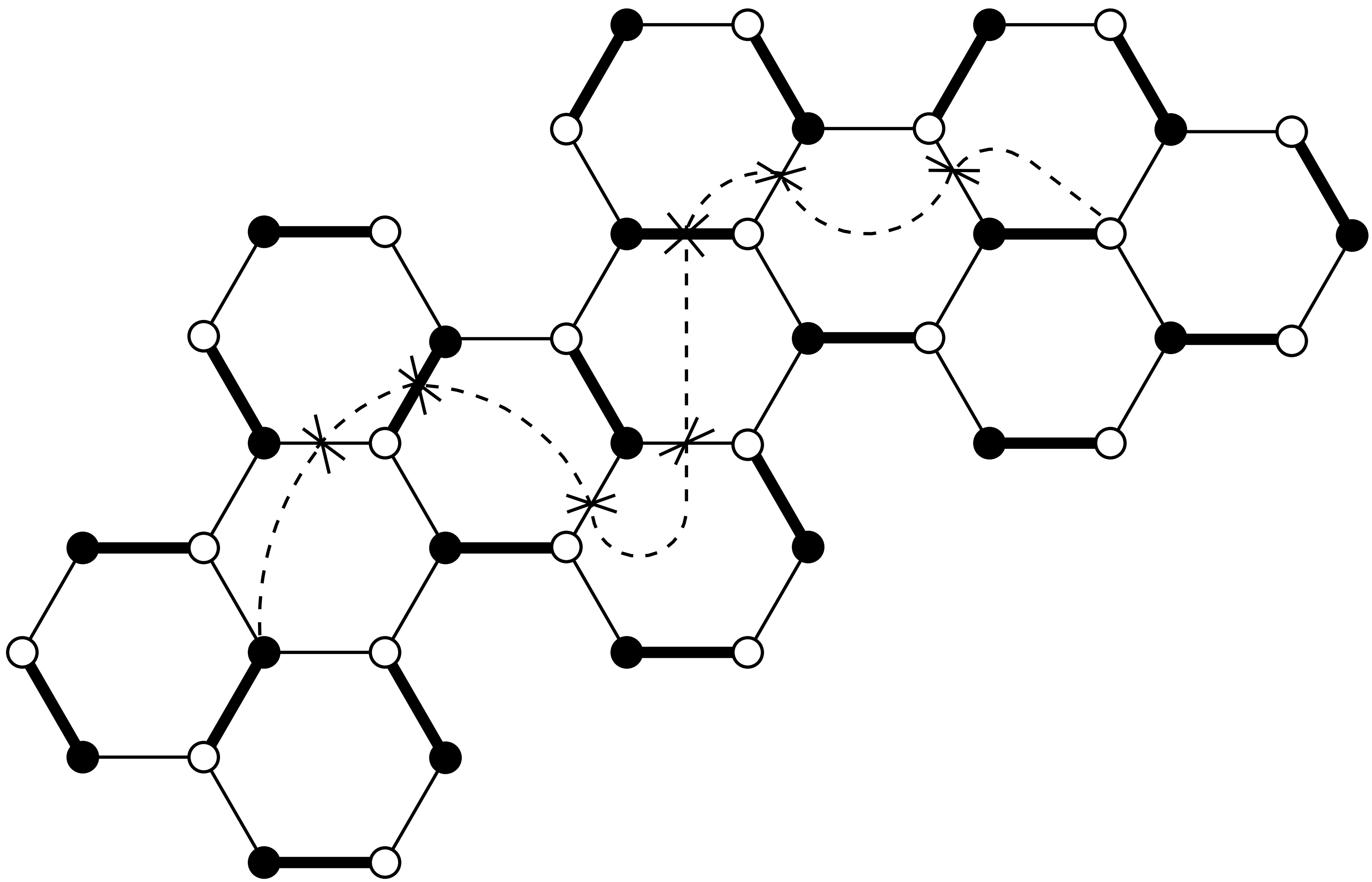


$h_0, h_1, h_2, h_3 = \boldsymbol{\mathscr{K}}$



$h_{012} =$ **non-$\mathscr{K}$**, $h_3 = \boldsymbol{\mathscr{K}}$.

Remark. To convert the non-Kasteleyn orientation back to Kasteleyn:



$$h_0 = h_1 = \cdots = h_{11} = -1.$$

Theorem. *Let* $X^{\mathscr{K}} \subset \overline{\mathcal{M}}_g \mid g=0$ *be multiedge embedding, for all number* $|\sigma_{2\xi-1}\sigma_{2\xi}|$ *of edges connecting* $\{\sigma_{2\xi-1}, \sigma_{2\xi}\} = \{\sigma_{k_{2\xi-1}}, \sigma_{k_{2\xi}}\}$, *then*

$$|\mathsf{Pf}(X^{\mathscr{K}})| = \mathcal{Z} \overset{def}{=\!=} \sum_{D} \prod_{D \,\cap\, k_{2\xi-1,\,2\xi}} \omega_{k_{2\xi-1,\,2\xi}}$$

where

$$\mathbf{Quot}(\mathbb{K}[D]) \ni \mathsf{Pf}(X^{\mathscr{K}}) = \frac{1}{n!}\frac{1}{2^n} \sum_{\sigma \in \mathcal{S}_{2n}} \mathit{sgn}(\sigma)\, X^{\mathscr{K}}_{\sigma_1\sigma_2} \cdots X^{\mathscr{K}}_{\sigma_{2n-1}\sigma_{2n}}$$

$$\mathit{sgn}(\sigma) = (-1)^{t(\sigma)} \;\mid\; t(\sigma) := (\sigma_1, \ldots, \sigma_{2n}) \longrightarrow (1, \ldots, 2n); \quad n \in \mathbb{N}$$

$$X^{\mathscr{K}}_{\sigma_{2\xi-1}\sigma_{2\xi}} = \sum_{k=1}^{|\sigma_{2\xi-1}\sigma_{2\xi}|} \varepsilon^{\mathscr{K}}_{\sigma_{k_{2\xi-1}}\sigma_{k_{2\xi}}} \omega_{\sigma_{k_{2\xi-1}}\sigma_{k_{2\xi}}}; \quad \forall\, \xi = 1, \ldots, n.$$

Proof. By $X^{\mathscr{K}} = m \times m \Longleftrightarrow m =$ even, then the positive-definite square $\det X^{\mathscr{K}} = \det(-(X^{\mathscr{K}})^T) = (-1)^m \det X^{\mathscr{K}} > 0$ of rational function of $X^{\mathscr{K}}_{ij}$.

Precisely, $X^{\mathscr{K}}_{i\,\pi_i} = -X^{\mathscr{K}}_{\pi_i\, i} \mid i \leqslant \pi_i \Longrightarrow$ sum of monomials in two partitions:

$$\left\{\begin{array}{l} \displaystyle\sum_{\substack{\pi \\ \Cap \\ \mathcal{S}_{2n}/(\mathcal{S}_n \times \mathcal{S}_2^n)}} (-1)^{t(\pi)} \prod_{i=1}^{2n} X^{\mathscr{K}}_{i\,\pi_i} \quad \left| \begin{array}{l} j = \pi^{-1}(\pi_i) \longleftrightarrow i \neq j \in \{1, \ldots, n\} \\ \quad \Longrightarrow X^{\mathscr{K}}_{i\,\pi_i} \equiv X^{\mathscr{K}}_{\pi_{2\xi-1}\pi_{2\xi}}, \ \forall\, \xi = 1, \ldots, n; \\ t(\pi) = \text{even (odd), for } n \text{ even (odd), resp.} \\ t(\pi) := (\pi_1, \ldots, \pi_{2n}) \longrightarrow (1, \ldots, 2n) \end{array} \right. \\ \qquad\qquad + \\ \displaystyle\sum_{\substack{\pi \\ \Cap \\ 2\cdot\left(\mathcal{S}_{\left\{\frac{(2n)!}{n!\,2^n}\right\}} \Big/ \left(\mathcal{S}_2 \times \mathcal{S}_{\left\{\frac{(2n)!}{n!\,2^n} - 2\right\}}\right)\right)}} (-1)^{t(\pi)} \prod_{i=1}^{2n} X^{\mathscr{K}}_{i\,\pi_i} \quad \left| \begin{array}{l} j = \pi^{-1}(\pi_i) \longleftrightarrow i \neq j \in \{1, \ldots, n\} \\ \Longrightarrow X^{\mathscr{K}}_{i\,\pi_i} \equiv X^{\mathscr{K}}_{\pi_{2\xi-1}\pi_{2\xi}}, \ \forall\, \xi = 1, \ldots, n; \\ t(\pi) = \text{odd (even), for } n \text{ even (odd),} \\ \text{respectively} \end{array} \right. \end{array}\right.$$

by second-index permutation (perhaps, first studied by Leibniz).

And, $t(\boldsymbol{\sigma}) := (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{2n}) \longrightarrow (1, \ldots, 2n)$ implies the quadratic:

$$\begin{cases} \displaystyle\sum_{\substack{\sigma = \widetilde{\sigma} \\ \Cap \\ \mathcal{S}_{2n} / (\mathcal{S}_n \times \mathcal{S}_2^n)}} (-1)^{t(\pi) + n + t(\sigma)} \left(\prod_{\xi=1}^{n} X^{\mathcal{K}}_{\sigma_{2\xi-1}\sigma_{2\xi}} \right)^2 & \Bigg| \; t(\pi) = \text{even (odd)}, \text{ for } n \text{ even (odd), resp.} \\ \qquad + \\ 2 \times \displaystyle\sum_{\substack{\left(\begin{smallmatrix} \sigma = \widetilde{\sigma} \,\neq\, \tau = \widetilde{\tau} \\ \Cap \\ \mathcal{S}_{2n} / (\mathcal{S}_n \times \mathcal{S}_2^n) \end{smallmatrix}\right) \\ \cong \\ \left(\mathcal{S}_{\left\{ \frac{(2n)!}{n!\,2^n} \right\}} \Big/ \left(\mathcal{S}_2 \times \mathcal{S}_{\left\{ \frac{(2n)!}{n!\,2^n} - 2 \right\}} \right) \right)}} (-1)^{t(\sigma) + t(\tau)} \prod_{\xi=1}^{n} X^{\mathcal{K}}_{\sigma_{2\xi-1}\sigma_{2\xi}} \prod_{\eta=1}^{n} X^{\mathcal{K}}_{\tau_{2\eta-1}\tau_{2\eta}} \end{cases}$$

$$= \left(\sum_{\sigma = \widetilde{\sigma}} (-1)^{t(\sigma)} \prod_{\xi=1}^{n} X^{\mathcal{K}}_{\sigma_{2\xi-1}\sigma_{2\xi}} \right)^2 = \mathsf{Pf}^2(X^{\mathcal{K}}) \quad \Bigg| \; t(\boldsymbol{\sigma}) := (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_{2n}) \longrightarrow (1, \ldots, 2n)$$

$\forall \min(\deg(X^{\mathcal{K}})) \geqslant n!\,a(X^{\mathcal{K}})\,b(X^{\mathcal{K}}) / \lfloor 2n-3 \rfloor!!;\ a, b \in \mathbb{R}^+;\ n \geqslant 2;\ \mathsf{Aut}(\mathcal{D}) \subseteq \mathcal{S}_{2n}.$

Thus, for all $|\sigma_{2\xi-1}\sigma_{2\xi}|$ edges connecting $\{\sigma_{2\xi-1},\sigma_{2\xi}\} = \{\sigma_{k_{2\xi-1}},\sigma_{k_{2\xi}}\}$:

$$\mathrm{Pf}(X^{\mathscr{K}}) = \sum_{\sigma=\widetilde{\sigma}} \mathrm{sgn}(\sigma) \prod_{\xi=1}^{n} \Bigg(\sum_{k=1}^{|\sigma_{2\xi-1}\sigma_{2\xi}|} \varepsilon^{\mathscr{K}}_{\sigma_{k_{2\xi-1}}\sigma_{k_{2\xi}}} \omega_{\sigma_{k_{2\xi-1}}\sigma_{k_{2\xi}}} \Bigg).$$

By the $\varepsilon^{\mathscr{K}}_{D}$ invariant of $\mathrm{Aut}(\mathscr{D})$,

$$\mathrm{Pf}(X^{\mathscr{K}}) = \sum_{D \,\Big|\, \substack{\sigma \\ \Cap \\ \mathrm{Aut}(\mathscr{D})/(\mathcal{S}_n \times \mathcal{S}_2^n)}} \underbrace{\mathrm{sgn}(\sigma) \prod_{\xi=1}^{n} \varepsilon^{\mathscr{K}}_{\sigma_{k_{2\xi-1}}\sigma_{k_{2\xi}}}}_{\text{fixed, } \forall\, \sigma \in \mathrm{Aut}(\mathscr{D})} \prod_{D \,\cap\, k_{2\xi-1,\,2\xi}} \omega_{k_{2\xi-1,\,2\xi}}$$

$$= \frac{1}{n!}\frac{1}{2^n} \sum_{D,\, \forall\, \sigma \in \mathrm{Aut}(\mathscr{D})} \underbrace{\varepsilon^{\mathscr{K}}_{D}}_{\text{fixed, } \forall\, \sigma \in \mathrm{Aut}(\mathscr{D})} \prod_{D \,\cap\, k_{2\xi-1,\,2\xi}} \omega_{k_{2\xi-1,\,2\xi}}$$

$$= \Bigg(\mathrm{sgn}(\sigma)\prod_{\xi=1}^{n}\varepsilon^{\mathscr{K}}_{\sigma_{k_{2\xi-1}}\sigma_{k_{2\xi}}}\Bigg)\sum_{D}\prod_{D\,\cap\,k_{2\xi-1,\,2\xi}}\omega_{k_{2\xi-1,\,2\xi}} = (\pm)\sum_{D}\prod_{D\,\cap\,k_{2\xi-1,\,2\xi}}\omega_{k_{2\xi-1,\,2\xi}} = \pm\,\mathcal{Z}.$$

That is,

$$\mathsf{Pf}(X^{\mathscr{K}}) = \sum_{\sigma\,\in\,\mathsf{Aut}(\mathscr{D})/(\mathcal{S}_n\times\mathcal{S}_2^n)} \mathsf{sgn}(\sigma)\prod_{\xi=1}^{n} X^{\mathscr{K}}_{\sigma_{2\xi-1}\sigma_{2\xi}} \quad \Big| \quad \sqrt{|\det X^{\mathscr{K}}|} = |\mathsf{Pf}(X^{\mathscr{K}})| = \mathcal{Z}$$

and, for all $\mathcal{S}_{2n}\backslash\mathsf{Aut}(\mathscr{D})$ monomials vanishing,

$$\mathsf{Pf}(X^{\mathscr{K}}) = \frac{1}{n!}\frac{1}{2^n}\sum_{\sigma\,\in\,\mathcal{S}_{2n}} \mathsf{sgn}(\sigma)\, X^{\mathscr{K}}_{\sigma_1\sigma_2}\cdots X^{\mathscr{K}}_{\sigma_{2n-1}\sigma_{2n}} \quad \Big| \quad |\mathsf{Pf}(X^{\mathscr{K}})| = \mathcal{Z}$$

differing only in orientation choice, independent of $\sigma \in \mathsf{Aut}(\mathscr{D})$. □

Theorem. *Observable is absolutely continuous iff $X^{\mathscr{K}}$ is non-singular.*

Proof.

$$\Big\langle \prod_{\ell=1}^{k} \sigma^{i_\ell j_\ell} \Big\rangle = \mathsf{Pf}\big((X^{\mathscr{K}})^{-1}_{\xi\eta}\big) \quad \Bigg| \quad \begin{array}{l} D \ni (i_1 j_1), \ldots, (i_k j_k);\;\; \xi,\eta = 1,\ldots,k \\ |\mathsf{Pf}(X^{\mathscr{K}})| = \text{partition function.} \end{array}$$ □

Theorem. Combinatorials i.e. exponentials reduce to cubic complexity.

Proof. $\mathsf{Pf}(\mathcal{A}X^{\mathscr{K}}\mathcal{A}^T) = \det(\mathcal{A})\mathsf{Pf}(X^{\mathscr{K}}) \longrightarrow \mathcal{O}(n^3)$ in diagonalization by skew symmetric Gaussian elimination, for spectrum analysis.

Remark. Recall mini-max contour deformation, critical point universality.

1.4 Grassmann (graded) integral

Definition. Grassmann (graded) algebra $\bigwedge^{\bullet} X^{\mathscr{K}}$, $\forall\, X^{\mathscr{K}}$ basis $(x_1, \ldots, x_{2n})$ is given by $2^{2n} = \sum_{k=0}^{2n}(\dim \bigwedge^k X^{\mathscr{K}}) = \sum_{k=0}^{2n}\binom{2n}{k}$ dimensional basis vectors:

$$\left\{ \begin{matrix} x_0 = 1;\ x_{\sigma_{k<}} = x_{\sigma_1} \otimes \cdots \otimes x_{\sigma_k} \mid x_{\sigma_\xi} \otimes x_{\sigma_\eta} + x_{\sigma_\eta} \otimes x_{\sigma_\xi} = 0; \\ \sigma_{k<} = (\sigma_1 \cdots \sigma_k) \mid \sigma_1 < \cdots < \sigma_k,\ \forall\, \sigma_1, \ldots, \sigma_k,\ k = 1, \ldots, 2n \end{matrix} \right\}.$$

Element is graded by

$$\bigwedge^{\bullet} X^{\mathscr{K}} \ni y(x) = y^{(0)} \oplus \sum_{i=1}^{2n} y^{(i)} x_i \oplus \bigoplus_{k=2}^{2n} \sum_{\tau \in \mathcal{S}_{\sigma_{k<}}} (-1)^{t(\tau)}\, y^{(\tau_1 \cdots \tau_k)}\, x_{\sigma_{k<}}$$

$$= \bigoplus_{k=0}^{2n} \sum_{\tau \in \mathcal{S}_{\sigma_{k<}}} y^{(\tau_1 \cdots \tau_k)} \bigotimes_{i=1}^{k} x_{\tau_i} \left| \begin{matrix} y^{([k=0])} = y^{(0)} \\ x_{\sigma_0} = x_0 = 1. \end{matrix} \right.$$

Multiplication $y_1(x)\, y_2(x)$ is given by

$$y_1^{(0)} y_2^{(0)} \oplus \sum_{i=1}^{2n} \left(y_1^{(0)} y_2^{(i)} + y_1^{(i)} y_2^{(0)} \right) x_i \oplus \frac{1}{2} \sum_{\sigma \in \mathcal{S}_{\sigma<}} \left(y_1^{(0)} y_2^{(\sigma_1 \sigma_2)} + \right.$$

$$\left. + y_1^{(\sigma_1)} y_2^{(\sigma_2)} - y_1^{(\sigma_2)} y_2^{(\sigma_1)} + y_1^{(\sigma_1 \sigma_2)} y_2^{(0)} \right) x_{\sigma_1} \otimes x_{\sigma_2} \oplus \cdots$$

Derivation. $\bigwedge^2 X^{\mathcal{K}} \ni w = \sum_{ij} X^{\mathcal{K}}_{ij} x_i \otimes x_j \implies \bigwedge^{2n} X^{\mathcal{K}} \ni w^n = \mathsf{Pf}(X^{\mathcal{K}}) x_{\sigma_{2n}<}$.

$$\bigotimes^k X^{\mathcal{K}} \longrightarrow \bigotimes^k X^{\mathcal{K}} :\ (-1)^{t(\sigma)} w_{\sigma_1} \wedge \cdots \wedge w_{\sigma_k} = \frac{1}{k!} \sum_{\tau \in \mathcal{S}_{\sigma_k <}} (-1)^{t(\tau)} \bigotimes_{i=1}^{k} w_{\tau_i}.$$

Definition. With respect to orientation $\theta \in \bigwedge^{2n} X^{\mathcal{K}} \cong \mathbb{R}$,

$$\int_{\bigwedge^{2n} X} f = f_\theta \qquad \Big|\ f = f_\theta\, \theta + \underbrace{\cdots}_{\text{lower order terms}}$$

by formal rule

$$\int \bigotimes_{i=1}^{2n} x_i \otimes \bigotimes_{i=1}^{2n} dx_i = (-1)^{\left(\sum_{i=1}^{2n-1} i\right)} \int \bigotimes_{i=1}^{2n} (x_i \otimes dx_i) = (-1)^{n(2n-1)}.$$

Derivation. For degenerate integral $\implies \deg(x) < \deg(dx)$,

$$\int \bigotimes_{i=1}^{k} x_{\sigma_i} \otimes dx = \begin{cases} (-1)^{t(\sigma)} & \text{if } k = 2n \\ 0 & \text{if } k < 2n \end{cases} \quad \Bigg|\ \begin{aligned} & dx = (-1)^{n(2n-1)} \textstyle\bigotimes_{i=1}^{2n} dx_i \\ & t(\sigma) := (\sigma_1, \ldots, \sigma_{2n}) \\ & \qquad \longrightarrow (1, \ldots, 2n). \end{aligned}$$

Derivation. $\theta = x_1 \otimes \cdots \otimes x_{2n}$ if (x_i) is basis of $X^{\mathcal{K}}$.

Theorem. *Let* $f(x) = \int_{\bigwedge^{\bullet} X^{\mathscr{K}}} \exp\left(\lambda_0 + \frac{1}{2}\left\langle x, X^{\mathscr{K}} x\right\rangle\right) dx$ *satisfy* $\mathsf{Pf}(X^{\mathscr{K}})$ *constraints*, *then* f *uniquely maximizes* $-\int_{\bigwedge^{\bullet} X^{\mathscr{K}}} (1/|f|)\log(1/|f|)\, dx$; *and*,

(i) $$\mathsf{Pf}(X^{\mathscr{K}}) = \int_{\bigwedge^{\bullet} X^{\mathscr{K}}} \exp\Big(\frac{1}{2}\sum_{ij} x_i X^{\mathscr{K}}_{ij} x_j\Big)\, dx$$

(ii) $$\mathsf{Pf}\begin{pmatrix} 0 & X^{\mathscr{K}} \\ -(X^{\mathscr{K}})^T & 0 \end{pmatrix} = \det(X^{\mathscr{K}})$$

(iii) $$(\mathsf{Pf}(X^{\mathscr{K}}))^2 = \det(X^{\mathscr{K}})$$

(iv) $$\frac{\partial}{\partial X^{\mathscr{K}}_{i_1 j_1}} \cdots \frac{\partial}{\partial X^{\mathscr{K}}_{i_k j_k}} \mathsf{Pf}(X^{\mathscr{K}}) = \mathsf{Pf}(X^{\mathscr{K}}) \cdot \mathsf{Pf}((X^{\mathscr{K}^{-1}})_{xy}) \Big|_{\substack{x = i_1, \ldots, i_k \\ y = j_1, \ldots, j_k}}.$$

Proof.

(i). Since all exponents except n vanish,

$$\int\limits_{\bigwedge^{\bullet} X^{\mathcal{K}}} \exp\Big(\frac{1}{2}\left\langle x, X^{\mathcal{K}} x\right\rangle\Big) dx \quad = \quad \frac{1}{n!}\frac{1}{2^n}\int\limits_{\bigwedge^{\bullet} X^{\mathcal{K}}} \left\langle x, X^{\mathcal{K}} x\right\rangle^n dx$$

where, precisely,

$$\int \left\langle x, X^{\mathcal{K}} x\right\rangle^n dx = \int \sum_{\sigma \in \mathcal{S}_n \times \mathcal{S}_2^n} X^{\mathcal{K}}_{i_1 j_1} \cdots X^{\mathcal{K}}_{i_n j_n} (x_{i_1} \otimes x_{j_1}) \otimes \cdots \otimes (x_{i_n} \otimes x_{j_n})\, dx \; =$$

$$= \sum_{\sigma \in \mathcal{S}_n \times \mathcal{S}_2^n} (-1)^{t(\sigma)} X^{\mathcal{K}}_{i_1 j_1} \cdots X^{\mathcal{K}}_{i_n j_n} \qquad \Big| \; t(\sigma) := (i_1, j_1, \ldots, i_n, j_n) \longrightarrow (1, \ldots, 2n).$$

Therefore, with "equality" of permutations $\sigma \in \mathcal{S}_n \times \mathcal{S}_2^n$,

$$\int\limits_{\bigwedge^{\bullet} V} \exp\Big(\frac{1}{2}\left\langle x, X^{\mathcal{K}} x\right\rangle\Big) dx \quad = \quad \mathsf{Pf}(X^{\mathcal{K}}).$$

Remark. II, III and IV follow from the latter integral formula.

(ii). Choosing splitting $X^{\mathcal{K}} = W^{\mathcal{K}} \oplus W^{\mathcal{K}}$ for block structure, where $X^{\mathcal{K}}$ is isomorphic to algebra (tensor product) generated by $u_i, v_i \,|\, i=1,\ldots,n$ with relations $u_i u_j = -u_j u_i$, $u_i v_j = -v_j u_i$, and $v_i v_j = -v_j v_i$:

$$
\begin{aligned}
(x_1, \ldots, x_{2n}) &= \\
&= \big(\underbrace{u_1, \ldots, u_n}_{\text{basis in } W^{\mathcal{K}}}, \underbrace{v_1, \ldots, v_n}_{\text{basis in } W^{\mathcal{K}}}\big).
\end{aligned}
$$

As a result,

$$
\left\langle x\,,\ \begin{pmatrix} 0 & X^{\mathcal{K}} \\ -(X^{\mathcal{K}})^T & 0 \end{pmatrix} x \right\rangle \;=\; 2\left\langle u, X^{\mathcal{K}} v\right\rangle
$$

i.e. need to prove

$$
\int\limits_{\bigwedge^n (W^{\mathcal{K}} \oplus W^{\mathcal{K}})} \exp\left(\left\langle u, X^{\mathcal{K}} v\right\rangle\right) du\, dv \;=\; \det(X^{\mathcal{K}}).
$$

(iii). Similar.

(iv). $$\int \exp\Big(\frac{1}{2}\langle x, X^{\mathscr{K}} x\rangle + \langle x, \eta\rangle\Big) dx \quad =$$

$$= \int \exp\Big(\frac{1}{2}\Big\langle x + X^{\mathscr{K}^{-1}}\eta\,,\ X^{\mathscr{K}}(x + X^{\mathscr{K}^{-1}}\eta)\Big\rangle \ - \ \frac{1}{2}\Big\langle \eta, X^{\mathscr{K}^{-1}}\eta\Big\rangle\Big) dx$$

$$= \exp\Big(-\frac{1}{2}\Big\langle \eta, X^{\mathscr{K}^{-1}}\eta\Big\rangle\Big)\,\mathsf{Pf}(X^{\mathscr{K}}).$$

$$\frac{\partial}{\partial X^{\mathscr{K}}{}_{i_1 j_1}} \cdots \frac{\partial}{\partial X^{\mathscr{K}}_{i_k j_k}}\ \mathsf{Pf}(X^{\mathscr{K}}) \ =$$

$$= \int \exp\Big(\frac{1}{2}\Big\langle x, X^{\mathscr{K}} x\Big\rangle\Big)\, x_{i_1} x_{j_1} \cdots x_{i_k} x_{j_k}\, dx$$

$$= \Big(\frac{\partial}{\partial \eta}\Big)^{2k} \int \exp\Big(\frac{1}{2}\Big\langle x, X^{\mathscr{K}} x\Big\rangle + \langle \eta, x\rangle\Big) dx.$$

Then, by Kullback-Leibler distance $\mathcal{D}(\cdot\|\cdot)$ and Jensen's inequality for any U,

$$-\mathcal{D}(U\|f)=\int\limits_{\bigwedge^\bullet X^{\mathscr{K}}}\frac{1}{|U|}\log\frac{(1/|f|)}{(1/|U|)}\leqslant\log\int\limits_{\bigwedge^\bullet X^{\mathscr{K}}}\frac{1}{|U|}\frac{(1/|f|)}{(1/|U|)}=\log\int\limits_{\bigwedge^\bullet X^{\mathscr{K}}}(1/|f|)=\log 1$$

i.e.

$$-\int\limits_{\bigwedge^\bullet X^{\mathscr{K}}}\frac{1}{|U|}\log\Big(\frac{1}{|U|}\Big)=-\int\limits_{\bigwedge^\bullet X^{\mathscr{K}}}\frac{1}{|U|}\log\Big(\frac{|f|}{|U|}\cdot\frac{1}{|f|}\Big)=-\mathcal{D}(U\|f)-\int\limits_{\bigwedge^\bullet X^{\mathscr{K}}}\frac{1}{|U|}\log\Big(\frac{1}{|f|}\Big)$$

$$\leqslant-\int\limits_{\bigwedge^\bullet X^{\mathscr{K}}}\frac{1}{|U|}\log\Big(\frac{1}{|f|}\Big)=-\int\limits_{\bigwedge^\bullet X^{\mathscr{K}}}\frac{1}{|U|}\log\left(\frac{1}{\left|\int_{\bigwedge^\bullet X^{\mathscr{K}}}e^{\lambda_0+\frac{1}{2}\langle x,X^{\mathscr{K}}x\rangle}dx\right|}\right)=-\int\limits_{\bigwedge^\bullet X^{\mathscr{K}}}\frac{1}{|f|}\log\Big(\frac{1}{|f|}\Big)$$

where the inequality is equality iff $U(x)=f(x)$ almost everywhere. □

Lemma. $\bigwedge^\bullet X^{\mathscr{K}}$ *graded identity, up to tensors on superalgebra* $M_{a,b}$ *minimal subfield, is isomorphic to kernel of either* $\mathbb{Q}$ *or prime-ordered field* $\mathbb{F}_{q=p^m}$.

Proof. ♡.

Theorem. *The ideal of* $M_{pr+qs,\,ps+qr}$ *is contained in ideal of* $M_{p,q}\otimes M_{r,s}$.

Proof. Follows from the prior lemma. □

Theorem. *Let* $X^{\mathscr{K}} \subset \overline{\mathcal{M}}_g \mid g=0$ *be bipartite multiedge embedding, then*

(i) $\mathcal{Z} = |\det(C_{X^{\mathscr{K}}})| \;\Big|\; C_{X^{\mathscr{K}}} = \mathbb{R}^{\mathscr{V}^{\circ}_{X^{\mathscr{K}}}} \hookleftarrow, \;\mathbb{R}^{V(X^{\mathscr{K}})} = \mathbb{R}^{\mathscr{V}^{\bullet}_{X^{\mathscr{K}}}} \oplus \mathbb{R}^{\mathscr{V}^{\circ}_{X^{\mathscr{K}}}} \hookleftarrow$

where $\hookleftarrow \implies$ nested.

(ii) $\left\langle \boldsymbol{\sigma}_{b_1 w_1} \cdots \boldsymbol{\sigma}_{b_k w_k} \right\rangle = \det\Big(\big((C_{X^{\mathscr{K}}})^{-1}\big)_{\widetilde{b}\, w}\Big) \;\left|\; \begin{array}{l} \widetilde{b} = \widetilde{b}_1, \ldots, \widetilde{b}_k \\ w = w_1, \ldots, w_k \end{array}\right.$

where $\widetilde{b}$ = white-vertex identified with b.

Proof.

(i). $X^{\mathscr{K}} \subset \overline{\mathcal{M}}_g \mid g=0$ implies

$$\mathcal{Z} \;=\; \boldsymbol{\varepsilon}_X^{\mathscr{K}} \int \exp\Big(\frac{1}{2}\sum_{ij} x_i\,(X_{ij}^{\mathscr{K}})\,x_j\Big)dx \qquad \left|\begin{array}{l} \boldsymbol{\varepsilon}_X^{\mathscr{K}} = (-1)^{\sigma}\,\boldsymbol{\varepsilon}^{\mathscr{K}}_{\sigma_1\sigma_2}\cdots\boldsymbol{\varepsilon}^{\mathscr{K}}_{\sigma_{2n-1}\sigma_{2n}} \\ 2n = |V(X^{\mathscr{K}})|. \end{array}\right.$$

$X^{\mathscr{K}} \subset \overline{\mathcal{M}}_g \mid g=0$ bipartite $\mathscr{V}_{X^{\mathscr{K}}} = \mathscr{V}^{\bullet}_{X^{\mathscr{K}}} \sqcup \mathscr{V}^{\circ}_{X^{\mathscr{K}}}$ implies

$$X^{\mathscr{K}} \;=\; \begin{pmatrix} 0 & B_{X^{\mathscr{K}}} \\ -(B_{X^{\mathscr{K}}})^T & 0 \end{pmatrix} \qquad \left|\begin{array}{l} B_{X^{\mathscr{K}}} :\; \mathbb{R}^{\mathscr{V}^{\circ}_{X^{\mathscr{K}}}} \longrightarrow \mathbb{R}^{\mathscr{V}^{\bullet}_{X^{\mathscr{K}}}} \\ \mathbb{R}^{V(X^{\mathscr{K}})} = \; \mathbb{R}^{\mathscr{V}^{\bullet}_{X^{\mathscr{K}}}} \oplus \mathbb{R}^{\mathscr{V}^{\circ}_{X^{\mathscr{K}}}} \\ \dim(\mathbb{R}^{\mathscr{V}^{\bullet}_{X^{\mathscr{K}}}}) = \dim(\mathbb{R}^{\mathscr{V}^{\circ}_{X^{\mathscr{K}}}}) = n \\ |V(X^{\mathscr{K}})| = 2n. \end{array}\right.$$

Identifying $V_{\bullet}(X^{\mathscr{K}})$, $V_{\circ}(X^{\mathscr{K}})$ via a diagram $\{b\} \sim \{w\}$ with "hole"

$$X^{\mathscr{K}} \;=\; \begin{pmatrix} 0 & C_{X^{\mathscr{K}}} \\ -(C_{X^{\mathscr{K}}})^T & 0 \end{pmatrix} \qquad \left|\begin{array}{l} \mathbb{R}^{V(X^{\mathscr{K}})} = \; \mathbb{R}^{\mathscr{V}^{\bullet}_{X^{\mathscr{K}}}} \oplus \mathbb{R}^{\mathscr{V}^{\circ}_{X^{\mathscr{K}}}} \hookleftarrow \\ C_{X^{\mathscr{K}}} = \; \mathbb{R}^{\mathscr{V}^{\circ}_{X^{\mathscr{K}}}} \hookleftarrow \\ \text{where } \hookleftarrow \;\Longrightarrow\; \text{nested} \end{array}\right.$$

i.e. $\mathcal{Z} \;=\; |\det(C_{X^{\mathscr{K}}})|.$ $\square$

(ii). Write

$$\left\langle \boldsymbol{\sigma}_{b_1 w_1} \cdots \boldsymbol{\sigma}_{b_k w_k} \right\rangle = \frac{\partial}{\partial\, w(b_1 w_1)} \cdots \frac{\partial}{\partial\, w(b_k w_k)} \ln \mathcal{Z}$$

$$= \det\Big(\big((C_{X\mathscr{K}})^{-1}\big)_{\widetilde{b}\,w}\Big) \;\Big|\begin{array}{l}\widetilde{b} = \widetilde{b}_1, \ldots, \widetilde{b}_k \\ w = w_1, \ldots, w_k\end{array}$$

where $\widetilde{b} =$ white-vertex identified with b. □

Remark. The "physical" meaning:

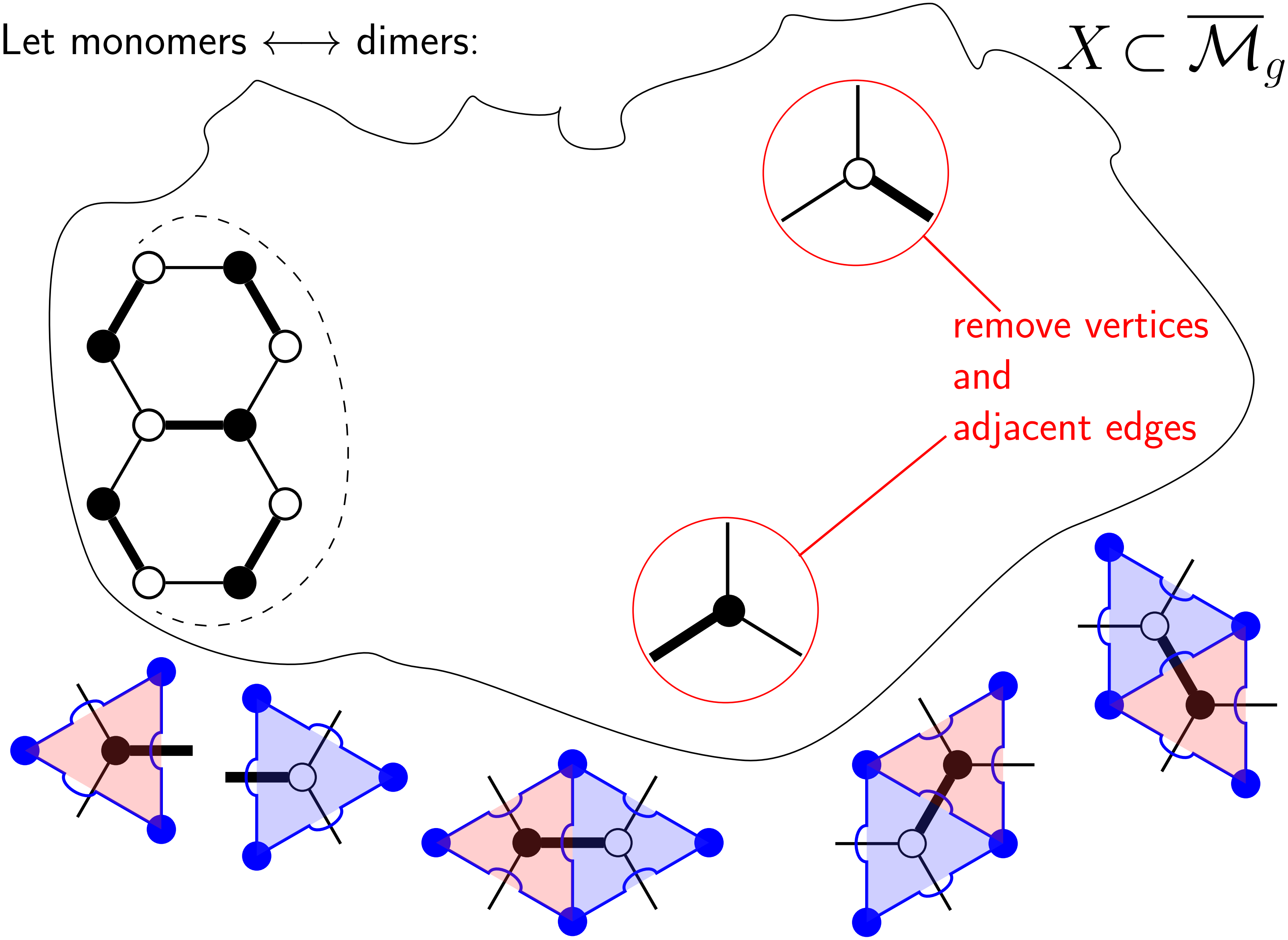
$$\left\langle \boldsymbol{\sigma}_{b_1 w_1} \cdots \boldsymbol{\sigma}_{b_k w_k} \right\rangle =$$

$$= \int \Psi^*_{b_1} \Psi_{w_1} \cdots \Psi^*_{b_k} \Psi_{w_k} \exp\big(\Psi^* C_{X\mathscr{K}}\, \Psi\big) d\Psi^* d\Psi \cdot \int \exp\big(\Psi^* C_{X\mathscr{K}}\, \Psi\big) d\Psi^* d\Psi$$

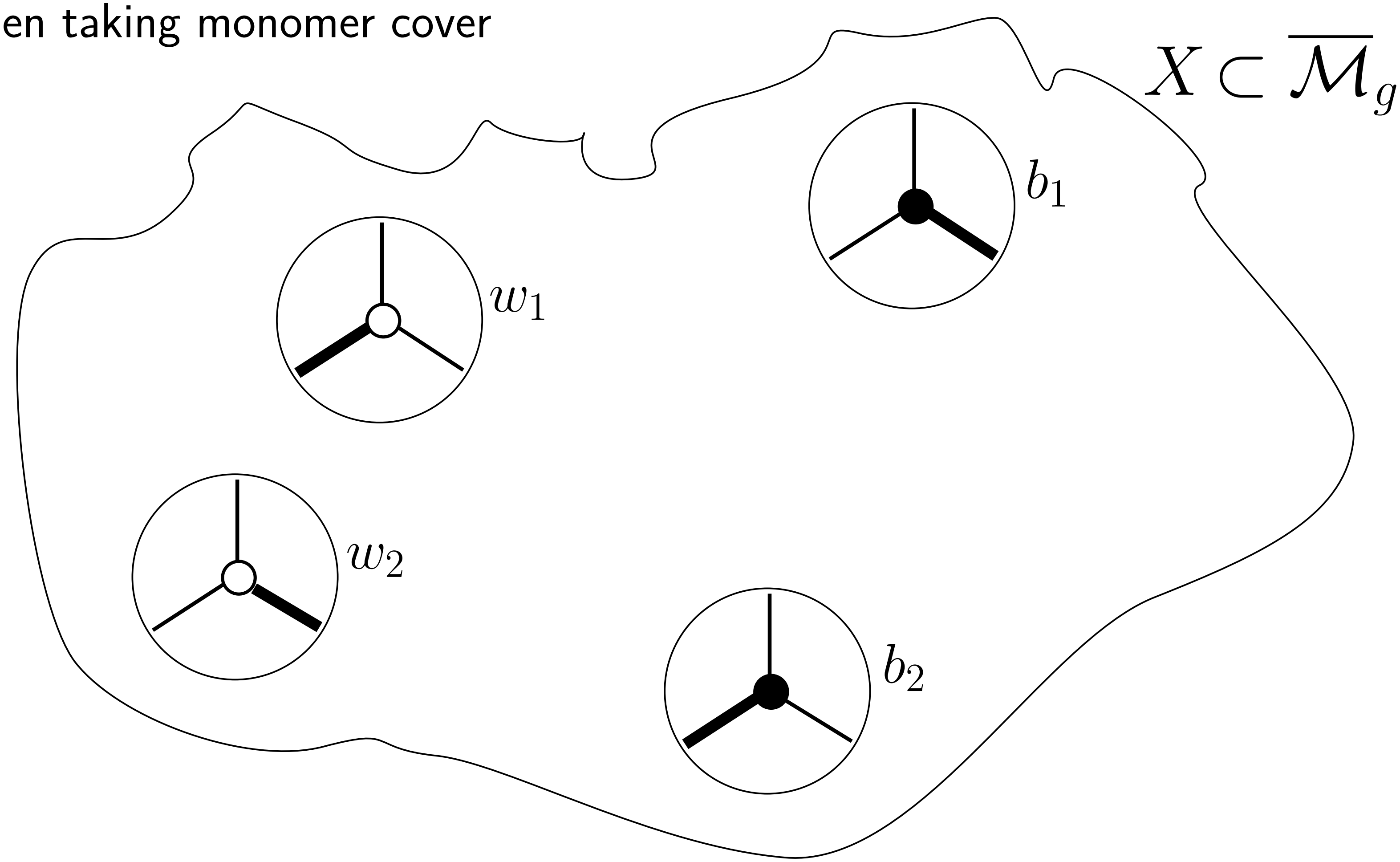
which corresponds to the free Fermionic observable.

Corollary (dimer-monomer problem).

Let monomers $\longleftrightarrow$ dimers:



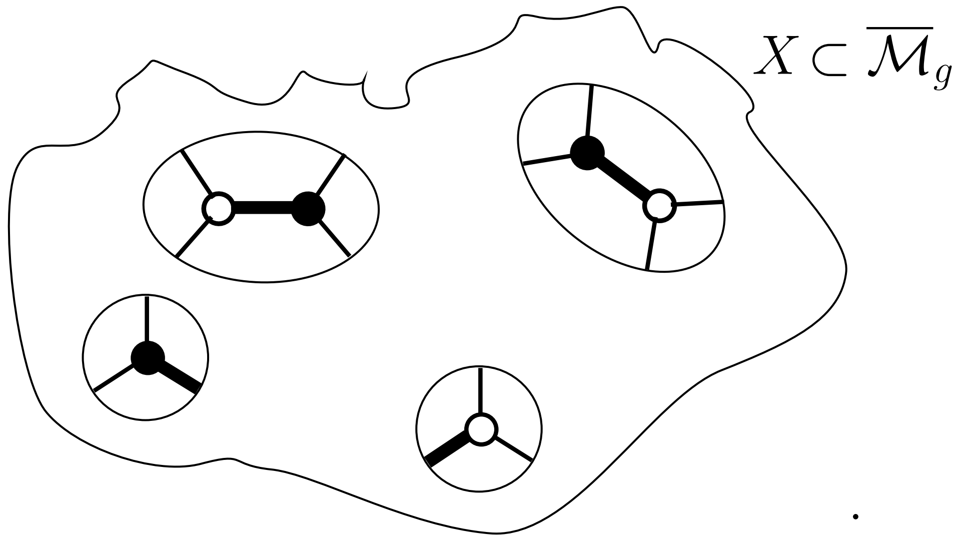
then taking monomer cover



the monomer-monomer observable is given by

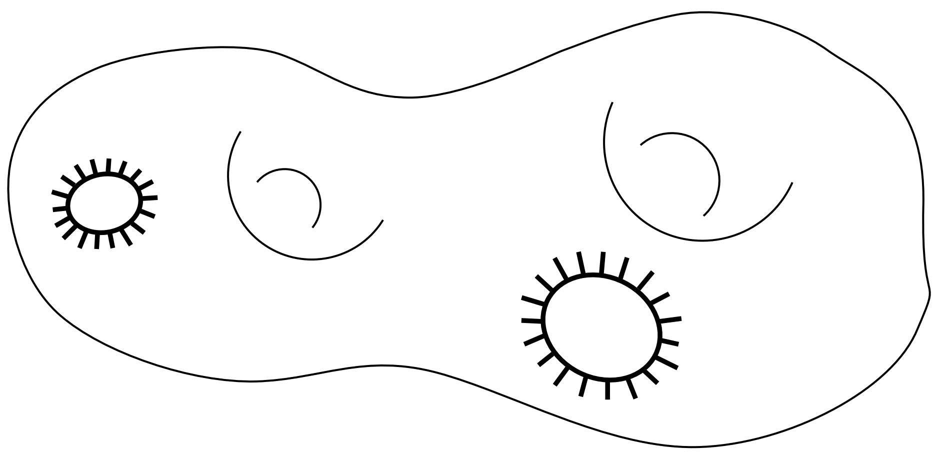
$$M_{b_1 \cdots b_n w_1 \cdots w_n} = \frac{\mathcal{Z}(X^{\mathscr{K}}_{b_1 \cdots b_n w_1 \cdots w_n})}{\mathcal{Z}(X^{\mathscr{K}})}$$

such that: adjacent monomers $(i_{b_\xi}, i_{w_\xi}) \Longrightarrow$ dimer $(i_{b_\xi}\, i_{w_\xi})$, $\forall\, \boldsymbol{\xi} \in D$:



.

and, $M_{b_1 \cdots b_n w_1 \cdots w_n}$, for all $|\{[\mathscr{K}]\}| = 2^{2g+2n-1}$, $2n = |$vertices$|$, is a special dimer case for nontrivial fundamental-group surfaces:



Proof. $\heartsuit$.

1.5 Generating function for $\mathscr{D}$ equivalence class $[\sigma]$

Given $\mathscr{D}$, an equivalence class $[\sigma] = \mathsf{Aut}(\mathscr{D})/(\mathcal{S}_n \times \mathcal{S}_2^n)$ order $|[\sigma]| = |\{\widetilde{\sigma}\}|$:

$$|\mathscr{D}| \Big/ \prod_{\xi=1}^{n} |\sigma_{2\xi-1}\sigma_{2\xi}| \leqslant |\{[\sigma]\}| = \Big(\exp\Big(n\ln 2 + \sum_{m=2}^{n} \ln m\Big)\Big)^{|\mathscr{D}|/\prod_{\xi=1}^{n}|\sigma_{2\xi-1}\sigma_{2\xi}|}$$

is given by two-variable generating function:

$$\sum_{\sigma=\widetilde{\sigma}} \prod_{\xi \in D(\sigma)} \sum_{\eta \,\in\, (\sigma(2\xi-1),\sigma(2\xi))} 1 \quad = \sum_{D(N_1,\dots,N_k)\,|\,(\sum_{\nu=1}^{k} N_\nu)=n} (\pm) \prod_{\nu=1}^{k} \omega_\nu^{N_\nu}$$

$\forall\, \eta$ connecting $\sigma(2\xi–1)$ and $\sigma(2\xi)$; $N_\nu = |\nu\text{-class dimers}|$; $\omega_1 = 1 = \omega_2$, $k = 2$.

Derivation I. Let $X \subset \overline{\mathcal{M}}_g =$ planar $M \times N$ square grid, where $\partial X =$ open.

$$|\{\widetilde{\sigma}(X;M,N)\}| = 2^{(\frac{MN}{2})} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\cos^2\Big(\frac{\pi i}{M+1}\Big) + \cos^2\Big(\frac{\pi j}{N+1}\Big)} \quad \Big| N = \text{even}$$

$$= |\{\widetilde{\sigma}(X;N,M)\}| \quad \Big| M = \text{even}$$

$$= 0 \quad \Big| MN = \text{odd}.$$

Show. ♡.

Derivation II. Let $X \subset \overline{\mathcal{M}}_g =$ cylindrical $M \times N$ square grid.

$$|\{\widetilde{\sigma}(X; M, N)\}| =$$

$$= 2^{(\frac{MN}{2})} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2(\frac{\pi(2i-1)}{M}) + \cos^2(\frac{\pi j}{N+1})} \quad \Bigg| N = \text{even}$$

$$= 2^{(\frac{MN}{2} - \frac{M}{2} + 1)} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2(\frac{\pi(2i-1)}{M}) + \cos^2(\frac{\pi j}{N+1})} \quad \Bigg| N = \text{odd}$$

$$= 0 \quad \Bigg| MN = \text{odd}.$$

Show. ♡.

Derivation III. Let $X \subset \overline{\mathcal{M}}_g =$ toroidal $M \times N$ square grid.

$$|\{\widetilde{\boldsymbol{\sigma}}(X; M, N)\}| =$$

$$= 2^{(\frac{MN}{2} - 1)} \left(\begin{array}{c} \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2(\frac{\pi(2i-1)}{M}) + \sin^2(\frac{2\pi j}{N})} \\ + \\ \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2(\frac{2\pi i}{M}) + \sin^2(\frac{\pi(2j-1)}{N})} \\ + \\ \prod_{i=1}^{M} \prod_{j=1}^{\frac{N}{2}} \sqrt{\sin^2(\frac{\pi(2i-1)}{M}) + \sin^2(\frac{\pi(2j-1)}{N})} \end{array} \right) \quad \Bigg| N = \text{even}$$

$$= |\{\widetilde{\boldsymbol{\sigma}}(X;\ N, M)\}| \quad \Big| M = \text{even}$$

$$= 0 \quad \Big| MN = \text{odd}.$$

Show. ♡.

Derivation IV. Let $X \subset \overline{\mathcal{M}}_g =$ planar 6×8 square grid, where $\partial X =$ open.

$$|\{\widetilde{\sigma}(X; M, N)\}| =$$

$$= 16777216 \, \Big(\frac{1}{4} + \cos^2(\frac{\pi}{7})\Big) \Big(\cos^2(\frac{\pi}{9}) + \cos^2(\frac{\pi}{7})\Big) \Big(\cos^2(\frac{\pi}{7}) + \cos^2(\frac{2\pi}{9})\Big) \times$$

$$\times \, \Big(\cos^2(\frac{\pi}{7}) + \sin^2(\frac{\pi}{18})\Big) \Big(\frac{1}{4} + \sin^2(\frac{\pi}{14})\Big) \Big(\cos^2(\frac{\pi}{9}) + \sin^2(\frac{\pi}{14})\Big) \times$$

$$\times \, \Big(\cos^2(\frac{2\pi}{9}) + \sin^2(\frac{\pi}{14})\Big) \Big(\sin^2(\frac{\pi}{18}) + \sin^2(\frac{\pi}{14})\Big) \Big(\frac{1}{4} + \sin^2(\frac{3\pi}{14})\Big) \times$$

$$\times \, \Big(\cos^2(\frac{\pi}{9}) + \sin^2(\frac{3\pi}{14})\Big) \Big(\cos^2(\frac{2\pi}{9}) + \sin^2(\frac{3\pi}{14})\Big) \Big(\sin^2(\frac{\pi}{18}) + \sin^2(\frac{3\pi}{14})\Big).$$

Show. $\heartsuit$.

Derivation V. Let $X \subset \overline{\mathcal{M}}_g =$ cylindrical 6×8 square grid.

$$|\{\widetilde{\sigma}(X;M,N)\}| =$$
$$= 5242880 \Big(\frac{1}{4} + \cos^2(\frac{\pi}{9})\Big)^2 \Big(1 + \cos^2(\frac{\pi}{9})\Big) \Big(\frac{1}{4} + \cos^2(\frac{2\pi}{9})\Big)^2 \times$$
$$\times \Big(1 + \cos^2(\frac{2\pi}{9})\Big) \Big(\frac{1}{4} + \sin^2(\frac{\pi}{18})\Big)^2 \Big(1 + \sin^2(\frac{\pi}{18})\Big).$$

Show. $\heartsuit$.

Derivation VI. Let $X \subset \overline{\mathcal{M}}_g =$ toroidal 6×8 square grid.

$$|\{\widetilde{\sigma}(X;M,N)\}| =$$
$$= 8388608 \Big[\frac{18225}{131072} + \cos^4(\frac{\pi}{8}) \Big(\frac{3}{4} + \cos^2(\frac{\pi}{8})\Big)^4 \sin^4(\frac{\pi}{8}) \Big(\frac{3}{4} + \sin^2(\frac{\pi}{8})\Big)^4 +$$
$$+ \Big(\frac{1}{4} + \cos^2(\frac{\pi}{8})\Big)^4 \Big(1 + \cos^2(\frac{\pi}{8})\Big)^2 \Big(\frac{1}{4} + \sin^2(\frac{\pi}{8})\Big)^4 \Big(1 + \sin^2(\frac{\pi}{8})\Big)^2 \Big].$$

Show. $\heartsuit$.

1.6 Partition as sum of Pfaffians

Lemma.

$$\mathcal{Z} = \frac{1}{2^g} \sum_{[\mathcal{K}]} \mathsf{Arf}(q_D^{\mathcal{K}}) \cdot \varepsilon^{\mathcal{K}}(D) \cdot \mathsf{Pf}(X^{\mathcal{K}}) \quad \left| \begin{array}{l} \pm 1 = \mathsf{Arf}(q) = \dfrac{1}{2^g} \displaystyle\sum_{\alpha \in \mathcal{H}^1} (-1)^{q(\alpha)} \\ 2^g = |\mathcal{H}^1(X^{\mathcal{K}};\, \mathbb{Z}_2)| \end{array} \right.$$

where

$[\mathcal{K}]$ = *all equivalence classes of Kasteleyn orientations,* 2^{2g} *in total*

$q_D^{\mathcal{K}}$ = *quadratic form on* $\mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)$, *corresponding to Kasteleyn orientation with respect to reference perfect matching* D

$$\varepsilon^{\mathcal{K}}(D) = (-1)^{\sigma} \varepsilon^{\mathcal{K}}_{\sigma_1\sigma_2} \cdots \varepsilon^{\mathcal{K}}_{\sigma_{n-1}\sigma_n} \quad \left| \begin{array}{l} \sigma \in \mathit{Aut}(D) \subseteq \mathit{Aut}(\mathscr{D}) \subseteq \mathcal{S}_n \\ D \cong [\sigma] = \mathit{Aut}(\mathscr{D}) \big/ (\mathcal{S}_{n/2} \times \mathcal{S}_2^{n/2}). \end{array} \right.$$

Proof. $\heartsuit$.

Theorem.

$$\mathcal{Z} = \frac{1}{2^g} \sum_{\mathfrak{T} \in S(\overline{\mathcal{M}}_g)} \mathsf{Arf}(q_{\mathfrak{T}}^{\mathcal{K}}) \cdot \mathsf{Pf}(X_{\mathfrak{T}}^{\mathcal{K}}) \quad \left| \quad \begin{array}{l} \pm 1 = \mathsf{Arf}(q) = \dfrac{1}{2^g} \displaystyle\sum_{\alpha \in \mathcal{H}^1} (-1)^{q(\alpha)} \\ 2^g = |\mathcal{H}^1(X^{\mathcal{K}};\, \mathbb{Z}_2)| \end{array} \right.$$

where

$\mathsf{Arf}(q_{\mathfrak{T}}^{\mathcal{K}}) :=$ *quadratic form* $q_{\mathfrak{T}}^{\mathcal{K}}$ *on* $\mathcal{H}^1(\overline{\mathcal{M}}_g; \mathbb{Z}_2)$ *for spin structure* $\mathfrak{T}$

$X_{\mathfrak{T}}^{\mathcal{K}} =$ *Kasteleyn matrix corresponding to spin structure* $\mathfrak{T}$

$S(\overline{\mathcal{M}}_g) =$ *set of all spin structures on* $\overline{\mathcal{M}}_g$.

Proof. $\heartsuit$.

Theorem. *Let* $X \subset \overline{\mathcal{M}}_g$ *be bipartite, such that*

height function $=$

$=$ *section of the non-trivial* $\mathbb{Z}$*-bundle.*

then

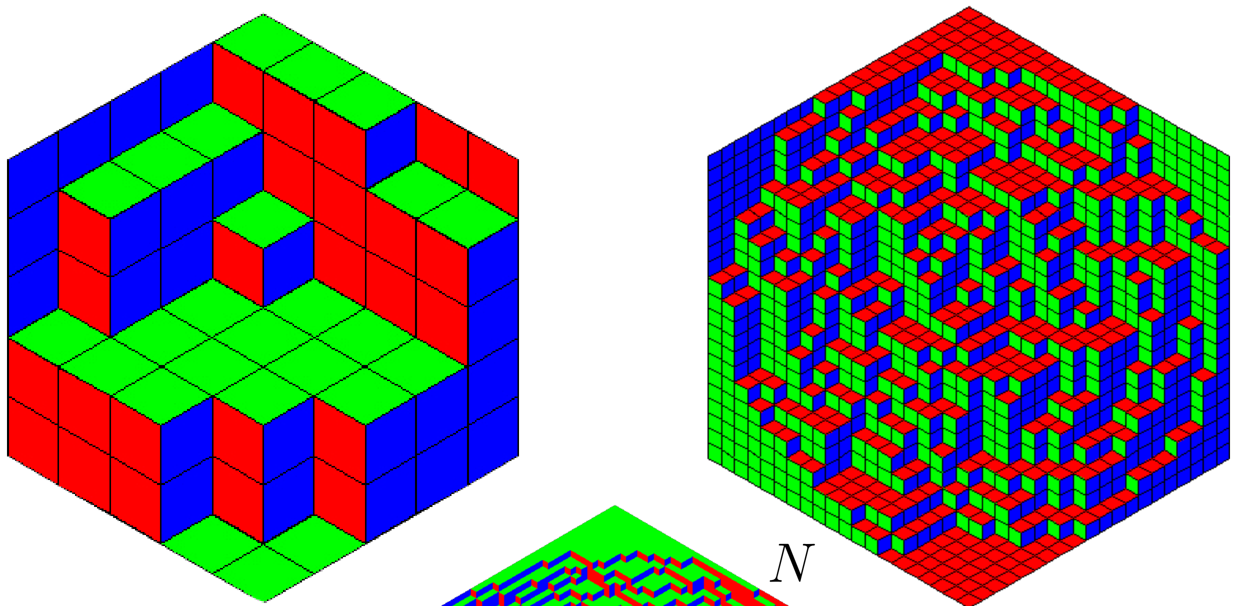
$$\mathcal{Z}\big(\mathcal{H}_{x_1},\ldots,\mathcal{H}_{x_g},\mathcal{H}_{y_1},\ldots,\mathcal{H}_{y_g}\big) = \\ = \sum_D \prod_{\ell \in D} \boldsymbol{\omega}(\ell) \prod_{i=1}^{g} \exp\Big(\sum_i \mathcal{H}_{x_i} \Delta_{x_i} h \; + \\ + \sum_i \mathcal{H}_{y_i} \Delta_{y_i} h \Big)$$

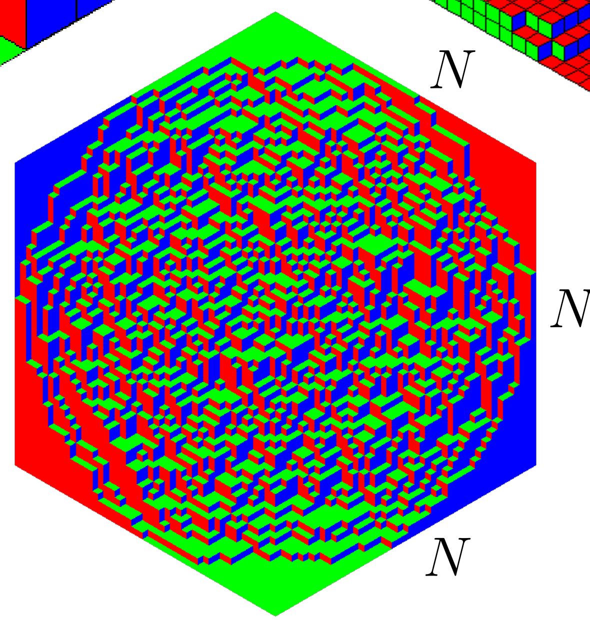
where $(x_1,\ldots,x_g,y_1,\ldots,y_g)$ *are fundamental cycles, and*

$\Delta_C h =$ *change in height function along* $\overline{\mathcal{M}}_g$ *noncontractible cycle* C*.*

Proof. $\heartsuit$.

1.7 Limits



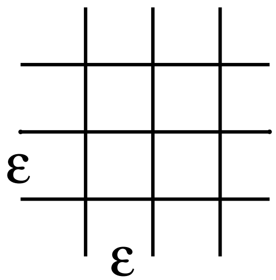


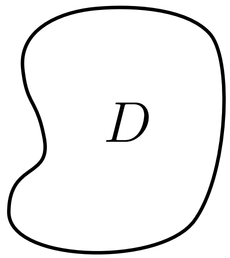
uniform measure

$$\text{Prob}(h) = \frac{1}{|\mathcal{H}_X|}$$

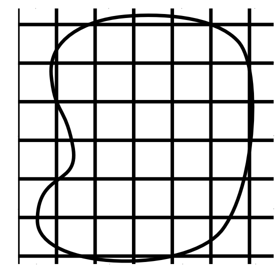
$$N \longrightarrow \infty \,.$$

Theorem (Schur process; Okounkov & R). Let $\varphi_\varepsilon : \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2 \mid D \subset \mathbb{R}^2$;





such that: $\begin{cases} \varepsilon \longrightarrow 0, \ |D_\varepsilon| \longrightarrow \infty \\ D_\varepsilon = \varphi_\varepsilon\big(\mathbb{Z}^2\big) \cap D \end{cases}$



.

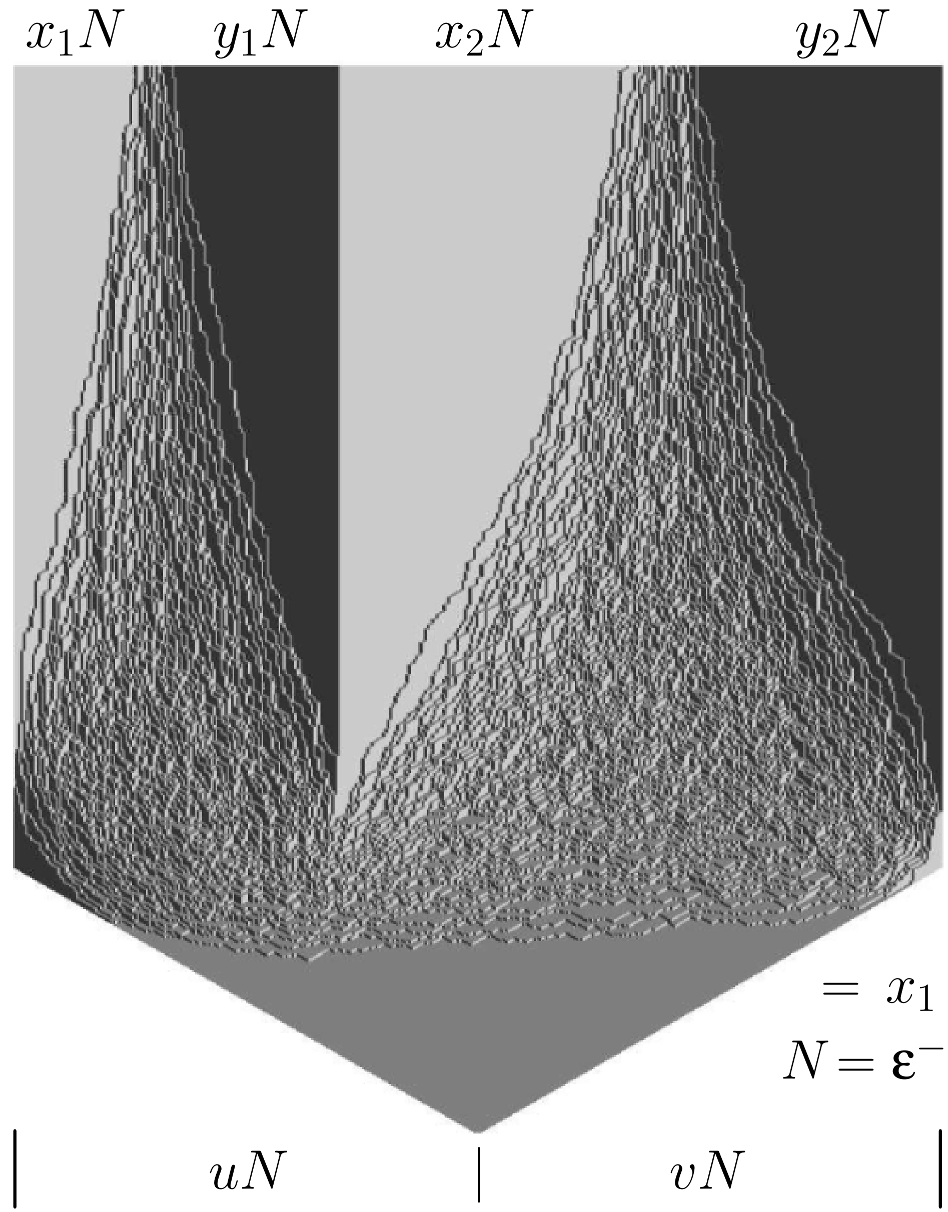
Then, for cube-stack with measure

$$\textit{Prob}(\pi) = \frac{\prod_t q_t^{\pi(t)}}{\sum_\pi \prod_t q_t^{\pi(t)}} \quad \left| \begin{array}{l} \pi \in \mathcal{H}_X \\ \pi \cong D, \end{array} \right.$$

there is existence of:

$$\textit{Thermodynamic limit } (|D_\varepsilon| \longrightarrow \infty) \ +$$
$$+ \ \textit{Scaling limit } (q = e^{-\varepsilon}, \ \varepsilon \longrightarrow +0).$$

Proof. $\heartsuit$.



where $u + v =$
$= x_1 + x_2 + y_1 + y_2;$
$N = \varepsilon^{-1},\ q = e^{-\varepsilon}.$

2 Vertex algebras

Points:

(i) Prove Grassmann kernel convergence for special genus g domain T^*

(ii) Obtain the $\mathbb{R}$ logarithmic scaling asymptotics by variational principle

(iii) State conjecture for the Green's function $\langle \cdot \rangle$ in large-deviation

2.1 Grassmann (graded) kernel

Pairing $\bigwedge^{\bullet} X^{\mathscr{K}*} \otimes \bigwedge^{\bullet} X^{\mathscr{K}} \longrightarrow \mathbb{R}$: $\sigma(k)> = (\sigma(1), \ldots, \sigma(k)) \big| \sigma(1) > \cdots > \sigma(k)$,

$$\langle \varphi(x^*), \psi(x) \rangle \stackrel{\text{def}}{=\!=} \varphi_0 \, \psi_0 + \sum_{k=1}^{2n} \varphi_k \psi_k + \sum_{k=1}^{2n} \sum_{\sigma(k)<} \varphi_{\sigma(k)\cdots\sigma(1)} \, \psi_{\sigma(1)\cdots\sigma(k)} \; =$$

$$= \; |\psi_0|^2 + \sum_{k=1}^{2n} \int_{\sigma(k)<} \left|\psi_{\sigma(1)\cdots\sigma(k)}\right|^2 d^{\,2n} x, \quad \forall \; |\psi|^2 \propto |\varphi|^2 \in \mathbb{R}$$

such that for the dual space, graded basis $x^*_{\sigma(k)>}$,

$$\bigwedge^{\bullet} X \ni \psi(x) = \psi_0 + \sum_{k=1}^{2n} \sum_{\sigma(k)<} \psi_{\sigma(k)<} \, x_{\sigma(k)<} \left| \bigwedge^{k} X^{\mathscr{K}} \ni \sum \psi_{\sigma(k)<} \, x_{\sigma(k)<} \right.$$

$$\bigwedge^{\bullet} X^{\mathscr{K}*} \ni \varphi(x^*) = \varphi_0 + \sum_{k=1}^{2n} \sum_{\sigma(k)>} \varphi_{\sigma(k)>} \, x^*_{\sigma(k)>} \left| \bigwedge^{k} X^* \ni \sum \varphi_{\sigma(k)<} \, x^*_{\sigma(k)>} \right.$$

where $\bigwedge^{\bullet} X^{\mathscr{K}*}$ is the dual graded algebra to $\bigwedge^{\bullet} X^{\mathscr{K}}$ generated by

$$\left\{ \begin{matrix} x_0 = 1; \;\; x_{\sigma_k<} = x_{\sigma_1} \otimes \cdots \otimes x_{\sigma_k} \; \big| \; x_{\sigma_\xi} \otimes x_{\sigma_\eta} + x_{\sigma_\eta} \otimes x_{\sigma_\xi} = 0; \\ \sigma_k< = (\sigma_1 \cdots \sigma_k) \mid \sigma_1 < \cdots < \sigma_k, \, \forall \, \sigma_1, \ldots, \sigma_k, \, k = 1, \ldots, 2n \end{matrix} \right\}.$$

Fixing integrals on $\bigwedge^\bullet X^{\mathscr{K}}$, $\bigwedge^\bullet X^{\mathscr{K}*}$, $\bigwedge^\bullet (X^{\mathscr{K}*} \otimes X^{\mathscr{K}})$ by choosing

$$x_1, \ldots, x_{2n} \in \bigwedge^{2n} X^{\mathscr{K}}, \quad x^*_{2n}, \ldots, x^*_1 \in \bigwedge^{2n} X^{\mathscr{K}*}$$

and

$$x^*_{2n}, \ldots, x^*_1, \, x_1, \ldots, x_{2n} \in \bigwedge^{2n} X^{\mathscr{K}*} \otimes \bigwedge^{2n} X^{\mathscr{K}}$$

then

$$\int \bigotimes_{i=1}^{\eta} x^*_{\sigma(i)} \bigotimes_{i=1}^{\eta} x_{\tau(i)} \, dx^* \, dx = \begin{cases} 0 & , \quad \eta \neq 2n \\ (-1)^{(\sigma + \tau + n(2n-1))} & , \quad \eta = 2n \end{cases}$$

$$\sigma : (\sigma(1), \ldots, \sigma(2n)) \longrightarrow (1, \ldots, 2n)$$

$$\tau : (\tau(1), \ldots, \tau(2n)) \longrightarrow (1, \ldots, 2n) \, .$$

Lemma.

$$\big\langle \varphi(x^*), \psi(x) \big\rangle = \int \exp\Big(\sum_i x^*_i x_i \Big) \, \varphi(x^*) \, \psi(x) \, dx^* \, dx \, .$$

Proof. $\heartsuit$.

Lemma. *Let* $Y^{\mathscr{K}} : X^{\mathscr{K}} \longrightarrow X^{\mathscr{K}}$ *by*

$$\begin{aligned} \psi_{Y^{\mathscr{K}}}(x) &= \sum_{\{i\}_<\,,\{j\}_<} x_{\{i\}_<} Y_{\{i\}_<\{j\}_<} \psi_{\{j\}_<} \\ &= \psi_0 \oplus Y\,\psi_1 \oplus Y^{\otimes 2}\psi_2 \oplus \cdots \end{aligned}$$

then

$$\psi_{Y^{\mathscr{K}}}(w) =$$

$$= \int \exp\Big(-x^* Y^{\mathscr{K}}\, w\Big)\ \exp(-x^* x)\ \psi(x) dx^*\, dx.$$

Proof. $\heartsuit$.

Lemma.

$$\int \exp\Big(-x^* Y^{\mathscr{K}}\, w\Big)\ \exp(-x^* x)\ \exp\Big(-W^{\mathscr{K}^*} W^{\mathscr{K}}\, x\Big)\ dx^* dx\ =$$

$$= \exp\Big(-w^* W^{\mathscr{K}} X^{\mathscr{K}}\, w\Big)\,.$$

Proof. $\heartsuit$.

Remark. Hence, $\exp(-w^* Y^{\mathscr{K}} w)$ is $Y^{\mathscr{K}}$ "integral kernel" acting on $\bigwedge^{2n} X^{\mathscr{K}}$.

2.2 Vertex operators

(i). The Fermionic Fock space F i.e. $\left\langle X_m^{\mathscr{K}} \right\rangle \in \mathbb{C}^{\mathbb{Z}+\frac{1}{2}}$ is given by

$$F = \left\{ X_{m_1}^{\mathscr{K}} \wedge X_{m_2}^{\mathscr{K}} \wedge \cdots \;\middle|\; \begin{array}{l} m_i \in \mathbb{Z} + \frac{1}{2} \\ m_{i+1} = m_i - 1 \\ i \gg 1 \end{array} \right\}.$$

(ii). The Clifford algebra is given by

$$Cl_{\mathbb{Z}} = \left\langle \psi_m \,,\, \psi_m^* \right\rangle \;\left|\; \begin{array}{l} m \in \mathbb{Z} + \frac{1}{2} \\ \psi_m \psi_{m'} + \psi_{m'} \psi_m \;=\; \psi_m^* \psi_{m'}^* + \psi_{m'}^* \psi_m^* \;=\; 0 \\ \psi_m \psi_{m'}^* + \psi_{m'}^* \psi_m \;=\; \delta_{m\,m'}\,. \end{array} \right.$$

(iii). The Clifford algebra acting on the Fock space F:

$$\psi_m \; x_{m_1} \wedge x_{m_2} \wedge \cdots \;=\; x_m \wedge x_{m_1} \wedge x_{m_2} \wedge \cdots$$

$$\psi_m^* \; x_{m_1} \wedge x_{m_2} \wedge \cdots \;=\; \sum_{i=1}^{\infty} (-1)^i \; \delta_{m_i,\,m} \; x_{m_1} \wedge \cdots \wedge \widehat{x_{m_1}} \wedge \cdots$$

(iv). The Heisenberg algebra is given by

$$\left\langle \alpha_n \right\rangle \quad \left| \begin{array}{l} n \in \mathbb{Z}\backslash\{0\} \\ [\alpha_n,\, \alpha_{n'}] \;=\; -n\,\delta_{n,\,-n'}\,. \end{array} \right.$$

(v). The Heisenberg algebra acting on the Fock space F:

- As part of Bose-Fermi correspondence in 1D:

$$\alpha_n \longmapsto \sum_{m \,\in\, \mathbb{Z}+\frac{1}{2}} \psi_{m+n}\,\psi_m^*\,.$$

- As operator in F:

$$\Big[\alpha_n\,,\,\psi_\xi\Big] = \psi_{\xi+n}\;\;, \quad \Big[\alpha_n\,,\,\psi_\xi^*\Big] = -\psi_{\xi-n}^*\,.$$

(vi). The vertex operators in F are given by

$$X_\pm^{\mathscr{K}}(x) = \exp\Big(\sum_{n=1}^{\infty} \frac{x^n}{n}\,\alpha_{\pm n}\Big) \quad \left| \begin{array}{l} \big(X_-^{\mathscr{K}}(x)v,\, w\big) = \\ \qquad = \big(v,\, X_+^{\mathscr{K}}(x)w\big) \\ \qquad = \big(X_+^{\mathscr{K}}(x)w,\, v\big). \end{array} \right.$$

(vii). The commutation relations are given by

$$\begin{aligned}
X_+^{\mathscr{K}}(x)\, X_-^{\mathscr{K}}(y) &= (1-x)\;\cdot\; X_-^{\mathscr{K}}(y)\, X_+^{\mathscr{K}}(x) \\
X_+^{\mathscr{K}}(x)\, \boldsymbol{\psi}(z) &= (1-z^{-1}\,x)^{-1}\;\cdot\; \boldsymbol{\psi}(z)\, X_+^{\mathscr{K}}(x) \\
X_-^{\mathscr{K}}(x)\, \boldsymbol{\psi}(z) &= (1-x\,z)^{-1}\;\cdot\; \boldsymbol{\psi}(z)\, X_-^{\mathscr{K}}(x) \\
X_+^{\mathscr{K}}(x)\, \boldsymbol{\psi}^*(z) &= (1-z^{-1}\,x)\;\cdot\; \boldsymbol{\psi}^*(z)\, X_+^{\mathscr{K}}(x) \\
X_-^{\mathscr{K}}(x)\, \boldsymbol{\psi}^*(z) &= (1-z\,x)\;\cdot\; \boldsymbol{\psi}^*(z)\, X_-^{\mathscr{K}}(x).
\end{aligned}$$

(viii). The eigenvectors are given by

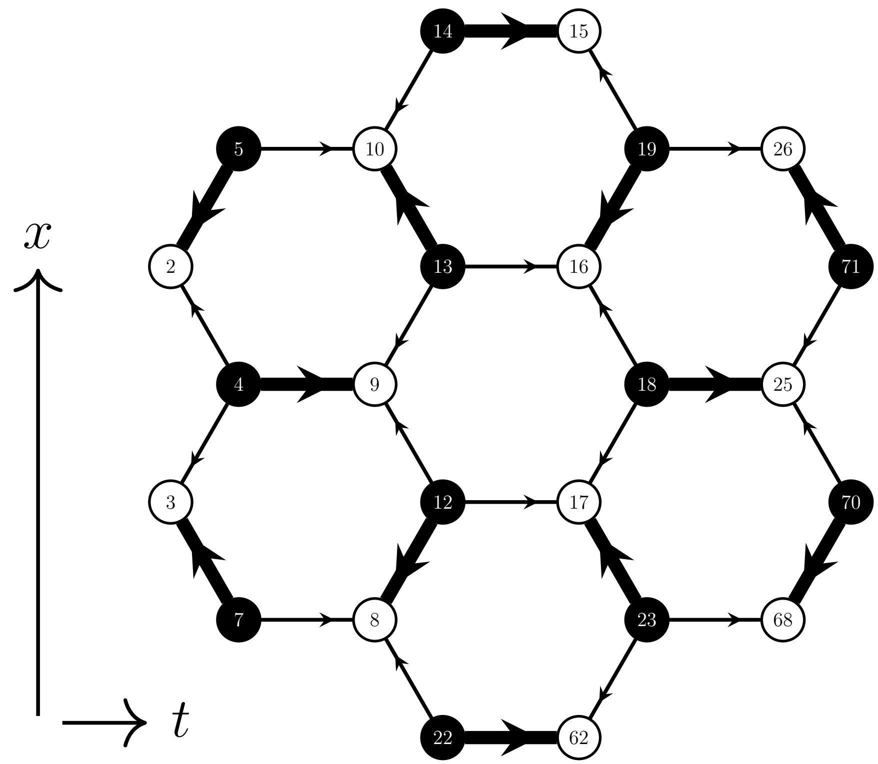
$$X_-^{\mathscr{K}}(x) \prod_i \boldsymbol{\psi}^*(w_i) \prod_j \boldsymbol{\psi}^*(z_j)\, v_0^{(n)} =$$

$$= \prod_i (1-x\,z_i)^{-1} \prod_j (1-x\,w_j) \prod_i \boldsymbol{\psi}^*(w_i) \prod_j \boldsymbol{\psi}^*(z_j)\, v_0^{(n)}$$

where $v_0^{(n)} = v_{n-\frac{1}{2}} \wedge v_{n-\frac{3}{2}} \wedge \cdots$

2.3 Fermionic Kasteleyn operators

For the one cube X^* of two-color tiles on bipartite hexagonal lattice X:



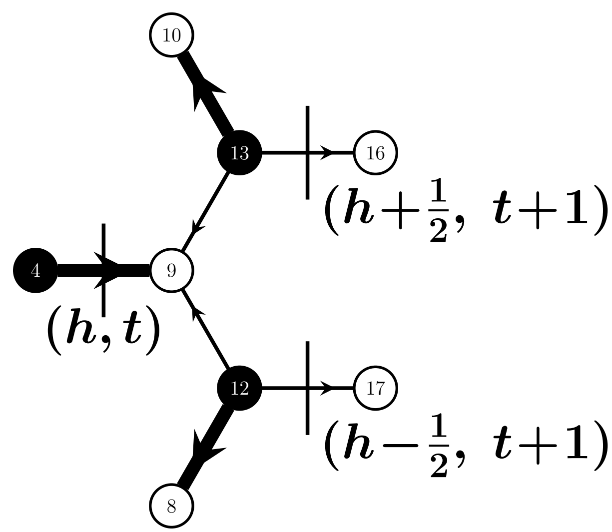
let the general parameterization for bipartite hexagonal lattice be given by

$$b(h,t) = (h,\ t - \tfrac{1}{2}),$$
$$w(h,t) = (h,\ t + \tfrac{1}{2}).$$

Kasteleyn matrix by the above-given $b \sim w$ diagram is then given by

$$K(h,t) \;=\; (h,t) \,-\, \big(h+\tfrac{1}{2},\, t+1\big) \,+\, y_{h,t}\,\big(h-\tfrac{1}{2},\, t+1\big).$$

Placing Fermions $x^*_{h,\,t}$, $x_{h,\,t}$ respectively at $b(h,t)$ and $w(h,t)$:



$$x^* K x \;=\; \sum_{h,\,t} x^*_{h,t}\, x_{h,t} \;-\; \sum_{h,\,t} x^*_{h+\frac{1}{2},\,t+1}\, x_{h,t} \;+\; \sum_{h,\,t} x^*_{h-\frac{1}{2},\,t+1}\, x_{h,t}\; y_{h,t}$$

$$=\; \sum_{t} \big(x^*_t\, x_t \;+\; x_t\, V\, x^*_{t+1} \;+\; x_t\, V^{-1}\, x_t\; x^*_{t+1}\big).$$

Theorem. *Assuming* $x_{h,t} = x_t,$ *analogous to the notation* $q_{h,t} = q_t,$

$[Diagram]$

Prob(π)
$\propto \prod_t q_t^{|\pi(t)|}$

the boundary conditions imply

$$\mathcal{Z} = \int \exp\big(x^* Y^{\mathcal{K}}\, x\big)\, dx^*\, dx \;=$$

$$= \Big\langle X_-^{\mathcal{K}}\big(x_{-\frac{1}{2}}\big) \cdots X_-^{\mathcal{K}}\big(x_{u_0+\frac{1}{2}}\big)\, X_+^{\mathcal{K}}\big(x_{\frac{1}{2}}\big) \cdots X_+^{\mathcal{K}}\big(x_{u_1+\frac{1}{2}}\big)\, v_0^{(0)}\,, \quad v_0^{(0)} \Big\rangle\;.$$

Proof (outline).

$$\int \cdots \exp\big(x^*_{t-1}\, x_{t-1}\big) \cdot \exp\big(x_{t-1}\, \big(V - V^{-1} X^{\mathcal{K}}_t\big)\, x^*_t\big) \cdot$$
$$\cdot \exp\big(x^*_t\, x_t\big) \cdot \exp\big(x_t\, \big(V - V^{-1} X^{\mathcal{K}}_t\big)\, x^*_{t+1}\big) \cdots \quad =$$
$$= \cdots \underbrace{\widetilde{\big(V - V^{-1} X^{\mathcal{K}}_{t-1}\big)}^{\,-1}}_{X^{\mathcal{K}}_+(x_t)} \cdot \underbrace{\widetilde{\big(V - V^{-1} X^{\mathcal{K}}_t\big)}^{\,-1}}_{X^{\mathcal{K}}_-(x_t)} \cdots$$

where $X^{\mathcal{K}}_+(x_t)$ and $X^{\mathcal{K}}_-(x_t)$ each depends on t such that

$$\widetilde{Y^{\mathcal{K}}} = Y^{\mathcal{K}}, \text{ where } V \hookleftarrow \text{ is lifted to } \textstyle\bigwedge^{\frac{\infty}{2}} V \;\Big|\; V = \bigoplus_{m \in \mathbb{Z}+\frac{1}{2}} \mathbb{C}\, v_h$$

under boundary conditions, etc. □

Remark. Direct proof exists combinatorially besides the Kasteleyn way.

Corollary.

$$\mathcal{Z} = \prod_{m=\frac{1}{2}}^{u_1 - \frac{1}{2}} \prod_{m' = u_0 + \frac{1}{2}}^{-\frac{1}{2}} \big(1 - x^-_{m'}\, x^+_m\big)^{-1}.$$

Theorem. (Okounkov & R., 2005).

$$\left\langle \boldsymbol{\sigma}_{(h_1 t_1)} \cdots \boldsymbol{\sigma}_{(h_k t_k)} \right\rangle = \det(K((t_i, h_i), (t_j, h_j)))_{1 \leqslant i,\, j \leqslant k}$$

$$K((t_i, h_i), (t_j, h_j)) =$$

$$= \frac{1}{(2\pi i)^2} \int\limits_{|z| < R(t_1)} \int\limits_{|z| < \widetilde{R}(t_2)} \frac{\Phi_-(z, t_1) \;\; \Phi_+(w, t_2)}{\Phi_+(z, t_1) \;\; \Phi_-(w, t_2)} \cdot$$

$$\cdot \frac{1}{z - w} \cdot z^{\left(-h_1 - B(t_1) - \frac{1}{2}\right)} \cdot w^{\left(h_2 - B(t_2) - \frac{1}{2}\right)} \, dz \, dw$$

where

$$\begin{array}{l|l} |w| < |z|,\ t_1 \geqslant t_2 & R(t) = \min\limits_{m > t}((x_m^+)^{-1}),\ \widetilde{R}(t) = \max\limits_{m < t}(x_m^-),\ B(t) = \frac{|t|}{2} - \frac{|t - u_0|}{2} \\ |w| > |z|,\ t_1 < t_2 & \Phi_+(z, t) = \prod\limits_{m > \max(t, \frac{1}{2})} (1 - z\, x_m^+),\ \ \Phi_-(z, t) = \prod\limits_{m < \max(t, -\frac{1}{2})} (1 - z^{-1}\, x_m^-). \end{array}$$

Proof. $\heartsuit$.

2.4 Thermodynamic limit with scaling

$[Diagram]$

$$\left.\begin{aligned} x_m^+ &= aq^m \\ x_m^- &= a^{-1}q^m \end{aligned}\right\}\text{assumed}$$

corresponding to $\text{Prob}(\pi) \propto q^{|\pi|}$.

Consider limit $\varepsilon \longrightarrow 0$, for $q = e^{-\varepsilon}$, $u_1 = \varepsilon^{-1} v_1$, $u_0 = \varepsilon^{-1} v_0$; fixed v_1, v_0:

$$\mathcal{Z} = \prod_{\substack{u_0 < n < 0 \\ 0 < m < u_1}} (1 - x_m^- x_n^+)^{-1} = \prod_{\substack{u_0 < n < 0 \\ 0 < m < u_1}} (1 - q^{m-n})^{-1}$$

$$\langle\, |\pi|\, \rangle = q \frac{\partial}{\partial q} \ln \mathcal{Z} = \varepsilon^{-3} \int_0^{u_1} \int_{u_0}^{0} \underbrace{\frac{s-t}{1-e^{t-s}}}_{\text{3D volume function}} \, ds\, dt + \cdots$$

where

$$\ln \mathcal{Z} = \varepsilon^{-2} \int_0^{u_1} \int_{u_0}^{0} \ln(\underbrace{1 - e^{-s+t}}_{\text{2D partition function}}) \, ds\, dt + \cdots$$

2.5 Graded (Grassmann) kernel asymptotics

Consider the limit $\varepsilon \longrightarrow 0$ where $t_i = \varepsilon^{-1}\tau_i,\ h_1 = \varepsilon^{-1}\chi_i,$ for fixed $\tau_i,\ \chi_i$:

$[Diagram]$ $(\tau_i,\ \chi_i)$ *in the bulk*

$$K((t_1,h_1),(t_2,h_2)) \longrightarrow$$

$$\longrightarrow \frac{1}{(2\pi i)^2} \int\limits_{C_z} \int\limits_{C_w} \exp\bigl(\varepsilon^{-1}(S(z,t_1,\chi_1) - S(z,t_2,\chi_2))\bigr) \cdot \cdot\ (zw)^{1/2}\ (z-w)^{-1}\ dz\ dw$$

where

$$S(z,t,\chi) =$$

$$= -\left(\chi + \frac{\tau}{2} - u_0\right) \ln \mathcal{Z} + \mathsf{Li}_2(ze^{-v_0}) + \mathsf{Li}_2(ze^{-v_1}) - \mathsf{Li}_2(z) - \mathsf{Li}_2(ze^{-\tau})$$

and

$$\mathsf{Li}_2(z) = \int\limits_0^z t^{-1} \ln(1-t)\ dt.$$

2.6 Critical point discriminants

$$\exp\left(\chi + \frac{\tau}{2}\right) \;=\; \frac{(1 - z e^{-v_0})(1 - z e^{-v_1})}{(1 - z)(1 - z e^{-\tau})}$$

gives quadratic equation, implying a discriminant for two real solutions or two complex-conjugate solutions, or a zero-discriminant.

$[Diagram]$

$$\boxed{\partial_\chi\, h_0(\tau, \chi) \;=\; \frac{1}{\pi}\, \mathsf{arg}(z_0)}$$

$$\left\langle \sigma_{(h,\,t)} \right\rangle \;=\; K((t, h),\, (t, h)) \;\longrightarrow\; \varepsilon\, \partial_\chi\, h_0(\tau, \chi).$$

2.7 Steepest descent

$$K((t_1,h_1),(t_2,h_2)) \;=\; -\frac{\varepsilon}{2\pi}\cdot\Big(\frac{\exp\{\varepsilon^{-1}(S_1(z_1)-S_2(w_2))\}}{(z_1-w_2)\sqrt{-w_2S_2''(w_2)}\;\sqrt{z_1S_1''(z_1)}} \;-$$

$$-\;\frac{\exp\{\varepsilon^{-1}(S_1(z_1)-S_2(\overline{w}_2))\}}{(z_1-\overline{w}_2)\sqrt{-\overline{w}_2S_2''(\overline{w}_2)}\;\sqrt{z_1S_1''(z_1)}}\;+\;c.\,c.\Big)\cdot(1+\mathcal{O}(1))$$

That is, for $\mathcal{H}_+ = \{z\in\mathbb{C},\ \mathsf{Im}\, z>0\} \mid z_0(\chi,\tau) =$ inner process, such that

$$z_1 \;= z_0(\chi_1,\tau_1)$$
$$w_2 \;= z_0(\chi,\tau),$$

$$K((t_1,h_1),(t_2,h_2)) \;=$$

$$=\;\frac{\varepsilon}{2\pi}\exp\{\varepsilon^{-1}(\mathsf{Re}(S(z_0(\chi_1,\tau_1)))-\mathsf{Re}(S(z_0(\chi_2,\tau_2)))))\}\cdot$$

$$\cdot\Big(\frac{\exp\{i\,\varepsilon^{-1}(\mathsf{Im}(S'(z_1))-\mathsf{Im}(S(w_2)))\}}{(z_1-w_2)}\;+$$

$$+\;\frac{\exp\{i\,\varepsilon^{-1}(\mathsf{Im}(S'(z_1))-\mathsf{Im}(S(\overline{w}_2)))\}}{(z_1-\overline{w}_2)}\;+\;c.\,c.\Big)\cdot(1+\mathcal{O}(1))\qquad(*).$$

Hence, (Kasteleyn-Fermions convergence to free Dirac-Fermions) solution:

$$\frac{1}{\sqrt{\varepsilon}}\psi_{\vec{x}} = \exp\{\varepsilon^{-1}\mathsf{Re}(S(z_0))\}\cdot\Big(\psi_+(z_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0))) + + \psi_-(\overline{z}_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0)))\Big)\cdot(1+\mathcal{O}(1))$$

$$\frac{1}{\sqrt{\varepsilon}}\psi^*_{\vec{x}} = \exp\{\varepsilon^{-1}\mathsf{Re}(S(z_0))\}\cdot\Big(\psi^*_+(z_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0))) + + \psi^*_-(\overline{z}_0)\,\exp(i\varepsilon^{-1}\mathsf{Im}(S(z_0)))\Big)\cdot(1+\mathcal{O}(1))$$

where

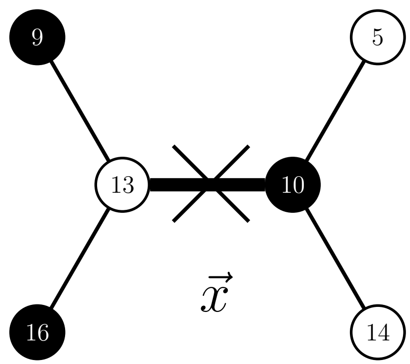
$$\mathbb{E}(\psi^*_\pm(z)\,\psi_\pm(w)) = \frac{1}{z-w}$$

$$\mathbb{E}(\psi^*_\pm(z)\,\psi_\mp(w)) = \mathbb{E}(\psi^*\,\psi^*) = \mathbb{E}(\psi\,\psi) = 0$$

such that $\psi^*_\pm(z)$, $\psi_\pm(w)$ are spinors:

$$\psi^*_\pm(z) = \psi^*_\pm(w)\sqrt{\frac{\partial w}{\partial z}}, \quad \psi_\pm(z) = \psi_\pm(w)\sqrt{\frac{\partial w}{\partial z}}.$$

Remark. The observable is given by:



$$\Big\langle (\sigma_{\vec{x}_1} - \langle \sigma_{\vec{x}_1} \rangle)\ (\sigma_{\vec{x}_2} - \langle \sigma_{\vec{x}_2} \rangle) \Big\rangle \ = \ K_{12} K_{21} \ =$$

$$= \frac{\varepsilon^2}{(2\pi)^2} \left(\frac{\frac{\partial z_1}{\partial x_1} \frac{\partial w_2}{\partial x_2}}{(z_1 - w_2)^2} \ - \ \frac{\frac{\partial z_1}{\partial x_1} \frac{\partial \overline{w}_2}{\partial x_2}}{(z_1 - \overline{w}_2)^2} \ + \ c.\,c. \right) \times$$

$$\times\ (1 + \mathcal{O}(1)).$$

In particular,

$$\sigma_{\vec{x}_1} - \langle \sigma_{\vec{x}_1} \rangle = \ \varepsilon\, \partial_x\, \varphi(z_0(\tau, x)) \ + \ \cdots \quad \Big| \ \varphi(z) = \text{Gaussian free field on } \mathcal{H}_+$$

such that the Green's function of Dirichlet problem on $\mathcal{H}_+$ is given by

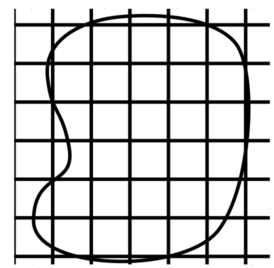
$$\langle \varphi(z)\, \varphi(w) \rangle = \ \frac{1}{2\pi} \ln \left| \frac{z - w}{z - \overline{w}} \right|$$

and, the Bose-Fermi correspondence is given by

$$\partial_x \varphi \ = \ :\widetilde{\psi}(z, \overline{z}) \ \ \widetilde{\psi}(z, \overline{z}): \ \ \cdots \ .$$

2.8 Scaling limit with Kasteleyn operator

Let $X = D_\varepsilon = \varphi_\varepsilon(L) \cap D$, for arbitrary lattice $L \mid A_G^{\mathcal{K}} =$ difference operator,



where $\varepsilon \longrightarrow 0$ in the asymptotics of the equation for $\mathcal{G}_{x,y}$ given by

$$(A_X^{\mathcal{K}})_x \cdot \mathcal{G}_{x,y} \; = \; \delta_{x,y}$$

Cases.

(i) Hexagonal lattice: Utilizes the weighted as above, for

$$q_t = e^{-\varepsilon f(t)}, \quad t = \frac{\tau}{\varepsilon}, \quad \varepsilon \longrightarrow 0.$$

Theorem. $\mathcal{G}_{x,y} =$ *same as* $(*)$, *with different* $z_0(\tau, x)$.

Proof. $\heartsuit$.

(ii) Periodic lattice: Utilizes variational principle.

2.9 Variational principle

(i). For the $N \times M$ torus

$$[Diagram]$$

$$\mathcal{Z}(H,V) = \sum_{D} \prod_{\ell \cap D} \omega_\ell \times \exp(H\Delta_a h_D + V\Delta_b h_D)$$

$$= \frac{1}{2}\left\{\mathsf{Pf}\big(A^{K_1}\big) + \mathsf{Pf}\big(A^{K_2}\big) + \mathsf{Pf}\big(A^{K_3}\big) - \mathsf{Pf}\big(A^{K_4}\big)\right\}$$

where $N, M \longrightarrow \infty$, for fixed $\frac{N}{M}$.

And, $\omega(\ell) = 1 \Longrightarrow$ eigenvalues of Kasteleyn matrices by Fourier transform.

Theorem. (McCoy & Wu, 1969; Kenyon & Okounkov, 2005).

$$\lim_{N,M\longrightarrow\infty} \frac{1}{NM} \ln \mathcal{Z}_{NM} = \oint\oint \ln|1+zw| \frac{dz}{z}\frac{dw}{w}$$

$$= f(H,V) \quad \Bigg| \begin{array}{l} |z| = e^H \\ |w| = e^V. \end{array}$$

(ii). Taking Legendre transform

$$\boldsymbol{\sigma}(s, t) \;=\; \max_{H,V}\left(H_s + V_t - f(H, V)\right)$$

then

$$\sum_D 1 \;=\; \sum_D \prod_D w\,(e) \;=\; \exp(MN\,\boldsymbol{\sigma}(s, t)\cdot(1+O(1)))$$

where

$$\frac{\Delta_a h_D}{M} = s, \quad \frac{\Delta_b h_D}{N} = t, \quad M, N \longrightarrow \infty, \quad \frac{N}{M} \text{ fixed.}$$

(iii). For domain

$$[Diagram]$$

$\Delta_a h = sM,\ \Delta_b h = tN.$

Theorem. (Cohn, Kenyon, & Propp, 2000).

$$\sum_D 1 \;=\; \exp(MN\,\boldsymbol{\sigma}(s, t)\cdot(1+\mathcal{O}(1)))$$

with the boundary conditions of height function h_D.

(iv). For domain

$$[Diagram] \qquad M_i \times N_j$$

$$\mathcal{Z}_{D_{\boldsymbol{\varepsilon}}} = \sum_{\left\{\substack{\text{values of} \\ \text{height functions} \\ \text{on} \\ \text{boundaries} \\ \text{between rectangles}}\right\}} \mathcal{Z}_{\underset{M_i}{\square N_j}} (h_{\text{bound}})$$

$$= \sum_{\{\Delta_x h,\, \Delta_y h\}_{ij}} \exp\left(\sum_{\underset{M_i}{\square N_j}} M_i\, M_j \;\boldsymbol{\sigma}\!\left(\frac{\Delta_x h}{M_i}, \frac{\Delta_y h}{N_j}\right) \right)$$

$$= \exp\left(\boldsymbol{\varepsilon}^{-2} \int_D \boldsymbol{\sigma}(\partial_x h_0,\, \partial_y h_0)\; dx\, dy\; (1 + \mathcal{O}(1)) \right)$$

where $h_0 =$ minimizer for

$$S[h] = \int_D \boldsymbol{\sigma}(\partial_x h_0,\, \partial_y h_0)\; dx\, dy.$$

Theorem. (Cohn, Kenyon, & Propp, 2000).

$$\lim_{\varepsilon \longrightarrow 0} \varepsilon^2 \ln \mathcal{Z}_{D_{\varepsilon}} = \int_D \sigma\big(\overrightarrow{\nabla} h_0\big) dx\, dy$$

for $0 < \partial_x h, \partial_y h < 1 \mid h_0 =$ *minimizer*

$h_0\big|_{\partial D} = b,$ *the boundary condition appearing in the limit* $\varepsilon \longrightarrow 0$

$$[Diagram]$$

for height function

$$h = \varepsilon^{-1} h_0 + \varphi = \varepsilon^{-1} (h_0 + \varepsilon\varphi)$$

with respect to $h_0 =$ *limit shape*, *and* $\varphi =$ *distribution (factor)*.

2.10 Physics way of the higher genus observable

$$S[h_0 + \varepsilon\varphi] = S[h_0] + \frac{\varepsilon^2}{2}\iint_D a^{ij}(x)\partial_i\varphi\,\partial_j\varphi\,d^2x$$

$$a^{ij}(x) = \partial_i\partial_j\,\varphi(s,t)\begin{cases} s = \partial_1 h_0 \\ t = \partial_2 h_0 \end{cases}$$

such that:

- Partition function equals

$$\mathcal{Z} = \exp(\varepsilon^{-2}S(h_0))\int \exp\left(\frac{1}{2}\iint_D a^{ij}(x)\partial_i\varphi\,\partial_j\varphi\,d^2x\right)D\varphi$$

where $D =$ scalar field with Riemannian metric induced by h_0;

- Correlation equals

$$\langle\varphi(x)\,\varphi(y)\rangle = \mathcal{G}(x,y)$$

where $\mathcal{G} =$ Green's function for $\Delta = \partial_i(a^{ij}\partial_j)$.

Conjecture. *$\mathcal{G}$ is same as obtained by asymptotics of Kasteleyn operators.*

Remark. The conjecture is theorem in certain cases.

(Chebotarev, Guskov, Ogarkov & Bernard, 2019). *For free-action or interaction Gaussian theory,*

$$\mathcal{S}[g, \bar{\varphi}] \equiv \frac{\mathcal{Z}[g, j = \hat{G}^{-1}\bar{\varphi}]}{\mathcal{Z}[j = \hat{G}^{-1}\bar{\varphi}]} =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\{ \prod_{a=1}^{n} \int d\Gamma_a e^{i\lambda_a \bar{\varphi}(x_a)} \right\} e^{-\frac{1}{2} \sum\limits_{a,b=1}^{n} \lambda_a \lambda_b G(x_a - x_b)} .$$

(Bernard, Guskov, Kalugin, Ivanov & Ogarkov, 2019). *In critical nonpolynomial phase theory,*

$$\mathcal{Z}\left[g; d\mu\right] = \int d\sigma_t \prod\!\!\!\!\!\!\int^{d\mu(x)} \left\| e^{\, f[\varphi(x);x]} \right\|_1 =$$

$$= \mathcal{C}_1\left[g; d\mu\right] \left\{ 1 + \frac{\pi(1-\eta)}{\Gamma^2(\frac{1}{4})} \int \frac{d\mu(x)}{\sqrt{2g(x)}} + \mathcal{O}\left[\frac{1}{g}\right] \right\} .$$

Conclusion: the higher genus observable yet

1. How to make (simulate) such pictures of perfect-matching mixture by:

   (i). Monte Carlo for $\exp(\propto 1000^2)$

   (ii). Sampling around most probable region by MCMC

2. How to describe such random surface invariant-limit analytically by:

   (i). Equipartition Pfaffian asymptotics with boundary conditions

   (ii). Variational principle: Minimizer functional in large deviation

References

[1] M. Bernard, V.A. Guskov, M.G. Ivanov, A.E. Kalugin, and S.L. Ogarkov. *Nonlocal scalar quantum field theory: functional integration, basis functions representation and strong coupling expansion*. Particles, 2(3):385–410, 2019.

[2] I.V. Chebotarev, V.A. Guskov, S.L. Ogarkov, and M. Bernard. *S-matrix of nonlocal scalar quantum field theory in basis functions representation*. Particles, 2(1):103–139, 2019.

[3] P.W. Kasteleyn. *Dimer statistics and phase transitions*. J. Math. Phys., 4:287–293, 1963.

[4] A. Okounkov and N. Reshetikhin. *Random skew plane partitions and the Pearcey process*. Comm. Math. Phys., 269:571–609, 2007.

Thank you!