An Extension of Tropical (Max-Plus) Algebra to Hilbert Space of Stochastic Time Series

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Abstract For a G-compact Sobolev boundary scheme of $C^{\infty}(M)$ singularities which is annulus-convergent- and disk-divergent-wise symmetric in height, given by length of a vertical trajectory, where G-composition map $\zeta : M^q_{\mathcal{X}}(\cdot,\cdot) \longmapsto M^q_{\mathcal{X}}(\cdot,\cdot)$ is q-Laplacian, $q \geq 5$, Lie-type decomposition on $(M;g)$, we investigate parameter estimation procedures and analysis of error distribution with stationarity in heuristic min $\wedge$ time extension of tropical max $\oplus$ plus $\otimes$ algebra on $\mathbb{R}$-separable Hilbert space of isomorphism of ζ-connected homotopy over M toric-covering in Riemann-tensor framework of G Chern-Pontryagin and Euler invariants. The isomorphism is diffusion-wise symmetric with a q-compact subquotient pseudocohomology $\big(H_c(M), d\big) \supset \big(M;g\big)^G / \mathcal{C}^{\infty}_g(\cdot) \cap \mathcal{C}^{\infty}_g(\mathcal{R}^q_{\max})$, and diffusions are allowed to vary in time to depend not only on solution state at t but also on sequence of probability distributions that converges in asymptotic at t.

Keywords: Tropical-Algebra, Hilbert-Space, Stochastic-time-series

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INTRODUCTION

By Sobolev M space of $\mathcal{C}^{\infty}_g(\cdot) \cap \mathcal{C}^{\infty}_g(\mathcal{R}^q_{\max}) \mid q \geq 5$, we mean freely coverable by k balls $B_{\partial_i}(r_i)_{i=1,\ldots,k}$ with radii (r_i) and operator $D^{\alpha}(X) := \dfrac{d^{\alpha}}{dX_1^{\alpha_1} \ldots dX_q^{\alpha_q}}(X)$ with degree $|\alpha| = \sum_{i=1}^{q} \alpha_i$ such that $M_{\mathcal{X}}(\cdot,\cdot) \xrightarrow{\sigma} M_{\mathcal{X}}(\cdot,\cdot) \ni X$ is extended tropical algebra or max-plus $\mathcal{X}_{\max} \overset{\text{def}}{=} \Big\{\mathcal{R}_{\max} \overset{\text{def}}{=} \mathbb{R} \cup \{-\infty\}, \oplus, \otimes, -\infty, 0\Big\}$ where the admissible, indempotent[a] additive $x \oplus y := x \vee y = \max\{x, y\}$ and piecewise-multiplicative $x \otimes y := x + y$ operations substitute conventional $x + y$ and $x \times y$ respectively, for $X \ni x, y$ complete in $M_{\mathcal{X}}(\cdot, \cdot)$. See Litvinov & Maslov [2005] for details.

Generally, the decomposition theory of a toric variety would consist of concrete description of the Hilbert space X^n of n-order decompositions. For the smooth toric variety completion on $(M; g)$, the combinatorial description would be given in terms of a corresponding algebra fan. That is, there is a canonical set of n-parameter decompositions over the affine line which, when the base is restricted to a fat point, spans X^n. And, there are a criterion for two (possibly singular) projective toric varieties to appear as special fibers in a common n-parameter flat family, with applications to mirror symmetry. In particular, by definition, the semiring $\mathcal{X} = (R, +_X, \times_X, 0_X, 1_X)$ is a nonempty set endowed with binary operations of "addition" $+_X$ and "multiplication" $\times_X$ such that: $+_X$ is commutative, associative, with zero element 0_X; $\times_X$ is associative, distributive over $+_X$, with unit element 1_X; and, 0_X is absorbing for $\times_X$. That is, the standard mapping $\zeta_X : D^{\alpha}(\mathcal{C}^{\infty}_{p+q}) \to \mathcal{X}(\mathcal{C}^{\infty}_{pq})$ given by $\mathcal{X}_{pq} := (\mathbb{R}, +, \times, 0, 1)$ is an instance of semiring from the viewpoint of real algebraic geometry. And, the semiring $\mathcal{X}$ becomes commutative if $\times_X$ is commutative, respectively indempotent if $+_X$ is indempotent.

Moreover, since conventional multiplication is commutative, if $\mathcal{X}_{\mathsf{st}}$ is commutative semiring, then $\mathcal{X}_{\mathsf{st}}$ is a well-known ring, even a field with respect to the operations $+$ and $\times$, although not idempotent. Hence, the following two properties hold: Max-plus algebra $\mathcal{X}_{\max} \overset{\text{def}}{=} \left\{\mathcal{R}_{\max} \overset{\text{def}}{=} \mathbb{R} \cup \{-\infty\}, \oplus, \otimes, -\infty, 0\right\}$ is both commutative semiring and idempotent semiring; and, min-plus algebra $\mathcal{X}_{\min} \overset{\text{def}}{=} \left\{\mathcal{R}_{\min} \overset{\text{def}}{=} \mathbb{R} \cup \{+\infty\}, \oplus', \otimes, +\infty, 0\right\}$ is by the same argument a commutative and idempotent semiring. In addition, the closure property, which is valid almost everywhere (a.e.) if valid for all, generalizes the field properties as follows: $0 = 1_{\mathcal{R}_{\max}} :=$ unit-element (for all multiplicative inverses x^{-1} and $-\infty = 0_{\mathcal{R}_{\max}} :=$ zero-element (for all additive inverses $-y$), since $x^{-1} \otimes 0 = x^{-1} = 0 \otimes x^{-1}$ and $-y \oplus -\infty = -y = -\infty \oplus -y, \ \forall\, x, y \in \mathcal{R}_{\max}$. In particular, $x^{-1} = -x,\ x^0 = 0,\ (-\infty)^0 = 0$, and $(-\infty)^n = -\infty\,|_{\,n>0}$, since

$$x^n \overset{\text{def}}{=} x^{\otimes n} = \underbrace{x \otimes \ldots \otimes x}_{n \text{ times}} = n \times x$$

while $(-\infty)^n\,|_{\,n<0}$ is indeterminate, that is, $-\infty$ has no inverse-element, since $-\infty$ is $\otimes$-self-absorbing: $x \otimes (-\infty) = (-\infty) \otimes x = -\infty,\ \forall\, x \in \mathcal{R}_{\max}$.

[a]indempotent for $x \oplus x = x$

Thus, under the max-plus operations, let V be a finite dimensional max-plus vector space over $\mathbb{C}$ (that is, similar to conventional vector space over $\mathbb{C}$ but with conventional operations substituted by max-plus operations). The set of linear operators $A : V \to V$ denoted by $\mathscr{L}(V)$ is defined by: $(A_1 + A_2)v = A_1 v + A_2 v,\ \ (A_1 A_2)v = A_1(A_2 v),\ \ (\lambda A)v = \lambda(Av)$ for $A_1, A_2 \in \mathscr{L}(V),\ \ v \in V,\ \ \lambda \in \mathbb{C}$. A basis $\{e_1, \cdots, e_n\}$ of the subspace with real dimension $n = \frac{1}{2} \cdot \log_2 |\{0,1\}^{2n}|$ gives an isomorphism of $\mathscr{L}(V)$ and the space of all $(n \times n)$ over $\mathbb{C}$ based on the following algebraic properties: Max-plus zero matrix is given by $-\infty_{m\times n} := (-\infty_{m\times n})_{ij} = -\infty|_{\forall\, i,j}$, with max-plus identity matrix $I_n := (I_n)_{ij} = \mathrm{mat}(a_{ij} = 0|_{i\,=\,j}\ \&\ a_{ij} = -\infty|_{i\,\neq\, j})$; and, max-plus matrix power is given by $A^0 = I_n$ with $A^n = A \otimes A^{n-1}$ for $n = 1, 2, \ldots$.

By distributivity, $x \otimes (y \oplus z) \ = \ (x \otimes y) \oplus (x \otimes z)$; by associativity $x \oplus (y \oplus z) \ = \ (x \oplus y) \oplus z,\ \ x \otimes (y \otimes z) = (x \otimes y) \otimes z$; and, by commutativity $x \oplus y = y \oplus x,\ \ x \otimes y = y \otimes x,\ \ \forall x, y, z \in \mathcal{R}_{\max}$. Following the indempotency of $\oplus$, existence of unit-element, zero-element, and the self-absorbing zero of $\otimes$, $\forall\, x \in \mathcal{R}_{\max},\ M_{\mathcal{X}}(\cdot,\cdot)$ is a complete M vector subspace, with respect to the tropical algebra defined on modular $\mathcal{X}$ lattice by:

$$(X \oplus Y)_{\xi\eta} = x_{\xi\eta} \oplus y_{\xi\eta} = \max(x_{\xi\eta}\,,\, y_{\xi\eta})$$

$$\big(X \otimes Z\big)_{\xi\eta} = \bigoplus_{\tau=1}^{n} \big(x_{\xi\tau} \otimes\, z_{\tau\eta}\big) = \max_{\tau}(x_{\xi\tau} + z_{\tau\eta})$$

for $X,\, Y \in\, \mathcal{R}_{\max}^{(\cdot)\times n}$, $Z \in \mathcal{R}_{\max}^{n\times(\cdot)}$; $\xi,\, \eta,\, \tau\, \in\, \mathbb{N}$, in addition to the logarithmic transform

$$x \quad \longmapsto \quad \Psi \;=\; h \log x$$

$h > 0$, which defines the map $\Gamma_h : \mathbb{R}^+ \longmapsto \mathbb{R}^+ \cup \{-\infty\}$. In particular, the tropical algebra becomes an extension of modular bootstrap technique on any Riemann manifold as follows: Think in barycentric coordinates of tetrahedral-lattice $\{X_{it\xi}\}$ random-metric couplings – including compact spheres and torus in $\big(\mathbb{Z}/N\mathbb{Z}\big)^2$ – with some colored balls at each vertex, and extra conditions to produce desired observations of appropriate subcell as quick as possible, while the set of observation is close to being linear, almost surely. And, the idea of success on ancillary is to bootstrap tropically along affine log-linear regression with moving average: $X_t = \sum_{i=1}^{q} \alpha_i X_{t-i} + \sum_{i=0}^{q} \beta_i \mathcal{E}_{t-i},\ \ \mathcal{E}_t \sim \mathcal{N}(0,1)$, including randomized inferences, with corrections made for the covariates defined with the admissible $\oplus$ and $\otimes$ in the completion $M_{\mathcal{X}}/\mathcal{R}_{\max}^{\{\oplus,\otimes,-\infty,0\}}$ of the following:

$$\left.\begin{aligned}\bigoplus_{i=1}^{q}(\beta_i\otimes\mathcal{E}_{t-i}) := (\beta_0\otimes\mathcal{E}_t)\bigoplus(\beta_1\otimes\mathcal{E}_{t-1})\bigoplus\cdots\\ \cdots\bigoplus(\beta_q\otimes\mathcal{E}_{t-q})\end{aligned}\right\}\;\Bigg|\;\xi\in\mathbb{N}^{+};\;\alpha_i,\,\beta_i\in\mathbb{R}$$

Affine Heuristics

Assuming $\|\cdot\|_\infty$ Sobolev-type evolution for a genus-g Riemann manifold $\Big(M;g\Big)$ with n marked poles[b], the first-order $\zeta_0^1 : M_{\mathcal{X}}(0,1)\longmapsto M_{\mathcal{X}}(0,0)$ is given by:

$$\begin{aligned}(1-\alpha B)X_n &= \mathcal{E}_n\\ X_n &= \alpha X_{n-1}+\mathcal{E}_n\\ G_n &= X_n-\mathcal{E}_n=\alpha X_{n-1}\end{aligned}$$

With $L^2(M)$ actions[c], the second-order $\zeta_0^2 : M_{\mathcal{X}}(0,2)\longmapsto M_{\mathcal{X}}(0,0)$ is given by:

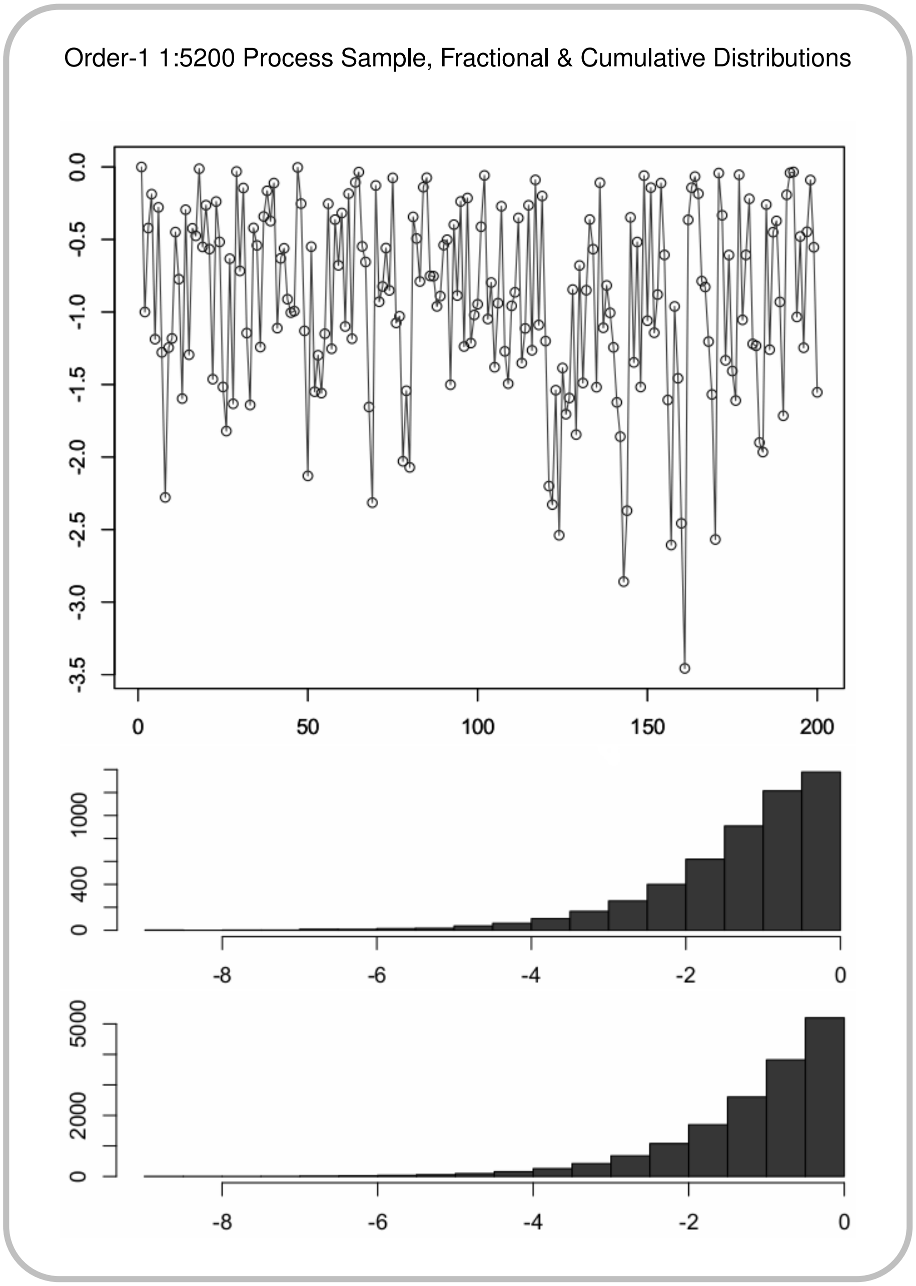
$$\begin{aligned}(1-\alpha_1 B-\alpha_2 B^2)X_n &= \mathcal{E}_n\\ X_n &= \alpha_1 X_{n-1}+\alpha_2 X_{n-2}+\mathcal{E}_n\\ G_n &= X_n-\mathcal{E}_n=\alpha_1 X_{n-1}+\alpha_2 X_{n-2}\end{aligned}$$

where $\alpha,\alpha_1,\alpha_2\in\mathbb{R}$ and $\mathcal{E}_n$ is $\mathcal{N}(0,1)$ for i.i.d $\{X_n,G_n\mid n\in\mathcal{T}=\mathbb{N}^{+}\}$.

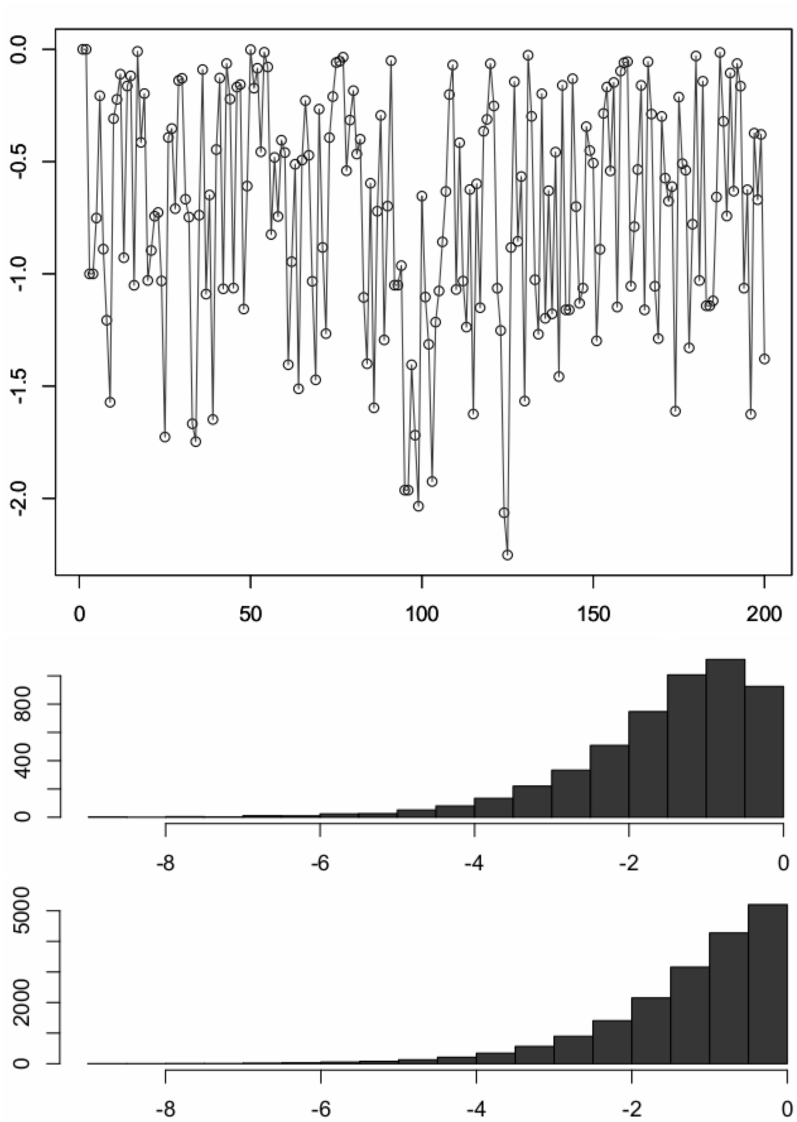
Thus, for i.i.d random exponential, fixing $\alpha=\alpha_1=\alpha_2=-1$ and $\mathcal{E}_n=-\delta_n$,

$$\left.\begin{aligned}&X_n=\alpha X_{n-1}+\mathcal{E}_n\\ &\text{goes to}\\ &X_n=\begin{cases}(-1\otimes X_{n-1})\\ \bigoplus\\ \mathcal{E}_n\end{cases}\end{aligned}\right\}\text{ and }\left\{\begin{aligned}&X_t=\alpha_1 X_{n-1}+\alpha_2 X_{n-2}+\mathcal{E}_n\\ &\text{goes to}\\ &X_n=\begin{cases}(-1\otimes X_{n-1})\\ \bigoplus\\ (-1\otimes X_{n-2})\\ \bigoplus\\ \mathcal{E}_n\end{cases}\end{aligned}\right. \tag{1}$$

[b] see Demidenko & Vaskevich [2006] for details
[c] of Sobolev-type evolution


Order-1 1:5200 Process Sample, Fractional & Cumulative Distributions
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
0
50
100
150
200
0
400
1000
-8
-6
-4
-2
0
0
2000
5000
-8
-6
-4
-2
0

Order-2 1:5200 Process Sample, Fractional & Cumulative Distributions



Recursion Inference and Stationarity

Where as stationarity in first-order exists for some convergent polynomial $f(B)$ (that is, where solution of $(1 - \alpha B) = 0$ is larger in absolute value than 1), stationarity in second-order exits for convergent weights α_i preserving observation of superimposed continuous $f(\mathbf{P}\{X \leq x\})$ terms on adjoined fractional bins.

By first-order, let $X := (\alpha \otimes X) \bigoplus (-\mathcal{E}_n)$; that is, $(-1 \otimes X) \bigoplus (-\mathcal{E}_n) := X$. Thus,

$$\begin{aligned}
\mathbf{P}\{(-1 \otimes X) \bigoplus (-\mathcal{E}_n) \leq t\} &= \mathbf{P}\{(-1 \otimes X) \leq t\} \bigoplus \mathbf{P}\{(-\mathcal{E}_n) \leq t\} \\
&= \mathbf{P}\{X \leq (t \otimes 1)\} \bigoplus \mathbf{P}\{\mathcal{E}_n \geq -t\} \\
&= \mathbf{P}\{X \leq (t \otimes 1)\} \bigoplus e^{\lambda t}.
\end{aligned}$$

For $k \in \mathbb{N}$, $F(\mathbf{P}\{X \leq t\}) = e^{\lambda t}$, $t \in [-1, 0]$, satisfies the set of procedures $\big\{F(t) \mid F_t \in F_k(t) = F_{k-1}(t \otimes 1)e^{\lambda t},\ t \in [-k, -k \otimes 1]\big\}$. Thus,

$$\begin{aligned}
F(\mathbf{P}\{X \leq t; k\}) &= e^{(\lambda(t \otimes 1))} e^{\lambda t} = e^{(2\lambda t \otimes \lambda)} && \text{t} \in [-2, -1] \\
&= e^{(2\lambda(t \otimes 1) \otimes \lambda \otimes \lambda t)} = e^{(3\lambda t \otimes 3\lambda)} && \text{t} \in [-3, -2] \\
&= e^{(3\lambda(t \otimes 1) \otimes 3\lambda \otimes \lambda t)} = e^{(4\lambda t \otimes 6\lambda)} && \text{t} \in [-4, -3] \\
&\vdots \\
F(\mathbf{P}\{X \leq t; k\}) &= \exp\left(k\lambda t \otimes \frac{k(k-1)\lambda}{2}\right) && \text{t} \in [-k, -k \otimes 1]
\end{aligned}$$

And, $f(\,\cdot\,) = F'(\,\cdot\,)$ implies:

$$f(\mathbf{P}\{X \leq t; k\}) = k\lambda \exp\left(k\lambda t \otimes \frac{k(k-1)\lambda}{2}\right) \qquad \text{t} \in [-k, -k \otimes 1]$$

By second-order, $(a_1 \otimes X_{n-1}) \wedge (a_2 \otimes X_{n-2}) \wedge (\mathcal{E}_n) := X_n$ with i.i.d exponentials $-\mathcal{E}_n$ and rate λ. Fixing ξ, with $\alpha_1 = -a_1$ and $\alpha_2 = -a_2$ implies:

$$F\Big(\mathbf{P}\Big\{X \leq x,\, y\Big\}\Big) = F\Big((x \otimes \alpha_1) \wedge y,\, x \otimes \alpha_2\Big) \times \exp\Big(\lambda\big(x \wedge 0\big)\Big).$$

Iterating with $2\alpha_1 > \alpha_2$ once:

$$F(\mathbf{P}\{X \leq x, y\}) =$$

$$= F\begin{pmatrix} \Big(x \otimes \alpha_2\Big) \wedge \Big(y \otimes \alpha_1\Big) \\ , \\ \Big(x \otimes \alpha_1 \otimes \alpha_2\Big) \wedge \Big(y \otimes \alpha_2\Big) \end{pmatrix} \times \exp\Big(\lambda\big(x \wedge 0 \otimes (x \otimes \alpha_1) \wedge y \wedge 0\big)\Big)$$

Thus, for any $k \in \mathbb{Z}^+$:

$$F\begin{pmatrix} \Big(x \otimes k\alpha_2\Big) \wedge \Big(y \otimes \alpha_1 \otimes (k-1)\alpha_2\Big) \\ , \\ \Big(x \otimes \alpha_1 \otimes k\alpha_2\Big) \wedge \Big(y \otimes k\alpha_2\Big) \end{pmatrix} =$$

$$= F\begin{pmatrix} \Big(x \otimes k\alpha_2 \otimes \alpha_2\Big) \wedge \Big(y \otimes \alpha_1 \otimes (k-1)\alpha_2 \otimes \alpha_2\Big) \\ \wedge \Big(x \otimes \alpha_1 \otimes k\alpha_2 \otimes \alpha_1\Big) \wedge \Big(y \otimes k\alpha_2 + \alpha_1\Big) \\ , \\ \Big(x \otimes k\alpha_2 \otimes \alpha_1 \otimes \alpha_2\Big) \wedge \Big(y \otimes \alpha_1 \otimes (k-1)\alpha_2 \otimes \alpha_1 \otimes \alpha_2\Big) \\ \wedge \Big(x \otimes \alpha_1 \otimes k\alpha_2 \otimes \alpha_2\Big) \wedge \Big(y \otimes k\alpha_2 \otimes \alpha_2\Big) \end{pmatrix}$$

$$\times \exp\left(\lambda \begin{pmatrix} \Big(x \otimes k\alpha_2\Big) \wedge \Big(y \otimes \alpha_1 \otimes (k-1)\alpha_2\Big) \wedge 0 \\ \otimes \\ \Big(x \otimes k\alpha_2 \otimes \alpha_1\Big) \wedge \Big(y \otimes \alpha_1 \otimes (k-1)\alpha_2 \otimes \alpha_1\Big) \\ \wedge\Big(x \otimes \alpha_1 \otimes k\alpha_2\Big) \wedge \Big(y \otimes k\alpha_2\Big) \wedge 0 \end{pmatrix}\right)$$

$$= F\begin{pmatrix} \Big(x \otimes (k \otimes 1)\alpha_2\Big) \wedge \Big(y \otimes \alpha_1 \otimes k\alpha_2\Big) \\ , \\ \Big(x \otimes \alpha_1 \otimes (k \otimes 1)\alpha_2\Big) \wedge \Big(y \otimes (k \otimes 1)\alpha_2\Big) \\ \wedge\Big(y \otimes (k \otimes 1)\alpha_2\Big) \end{pmatrix}$$

$$\times \exp\left(\lambda \begin{pmatrix} \Big(x \otimes k\alpha_2\Big) \wedge \Big(y \otimes \alpha_1 \otimes (k-1)\alpha_2\Big) \wedge 0 \\ + \\ \Big(x \otimes \alpha_1 \otimes k\alpha_2\Big) \wedge \Big(y \otimes k\alpha_2\Big) \wedge 0 \end{pmatrix}\right)$$

Putting these observations together:

$$F\big(\mathbf{P}\{X \le x, y; k\}\big) =$$

$$= \exp\left(\lambda\begin{pmatrix} x \wedge 0 \\ \otimes \\ (x \otimes \alpha_1) \wedge y \wedge 0 \end{pmatrix}\right)$$

$$\times \prod_{k=1}^{\infty} \exp\left(\lambda\begin{pmatrix} \big(x \otimes k\alpha_2\big) \wedge \big(y \otimes \alpha_1 \otimes (k-1)\alpha_2\big) \wedge 0 \\ + \\ \big(x \otimes \alpha_1 \otimes k\alpha_2\big) \wedge \big(y \otimes k\alpha_2\big) \wedge 0 \end{pmatrix}\right)$$

$$= \exp\left(\lambda\left(\begin{matrix} \big(x \wedge 0 \otimes (x \otimes \alpha_1) \wedge y \wedge 0\big) \\ + \\ \displaystyle\sum_{k=1}^{\infty} \begin{pmatrix} \big(x \otimes k\alpha_2\big) \wedge \big(y \otimes \alpha_1 \otimes (k-1)\alpha_2\big) \wedge 0 \\ + \\ \big(x \otimes \alpha_1 \otimes k\alpha_2\big) \wedge \big(y \otimes k\alpha_2\big) \wedge 0 \end{pmatrix} \end{matrix}\right)\right)$$

where the infinite series actually terminates.

By further observation of the piecewise intervening phenomenon

$$F\big(\mathbf{P}\{X \le x, +\infty; k\}\big) =$$

$$= \exp\left(\lambda\left(\begin{matrix} \big(x \wedge 0 \otimes (x \otimes \alpha_1) \wedge 0\big) \\ \bigotimes \\ \displaystyle\sum_{k=1}^{\infty} \begin{pmatrix} \big(x \otimes k\alpha_2\big) \wedge 0 \\ + \\ \big(x \otimes \alpha_1 \otimes k\alpha_2\big) \wedge 0 \end{pmatrix} \end{matrix}\right)\right)$$

$$= \exp\left(\lambda\left(\sum_{k=1}^{\infty} \begin{pmatrix} \big(x \otimes k\alpha_2\big) \wedge 0 \\ + \\ \big(x \otimes \alpha_1 \otimes k\alpha_2\big) \wedge 0 \end{pmatrix}\right)\right)$$

In addition, further distinguishing subsequent events by observing significant steps of previous evolutions:

$$F\big(\mathbf{P}\{X \leq +\infty, y; k\}\big) \ =$$

$$= \ \exp\left(\lambda\left(\begin{array}{c}\Big(y \wedge 0\Big) \\ \\ \bigotimes \\ \\ \displaystyle\sum_{k=1}^{\infty}\left(\begin{array}{c}\Big(y \otimes \alpha_1 \otimes (k-1)\alpha_2\Big) \wedge 0 \\ + \\ \Big(y \otimes k\alpha_2\Big) \wedge 0\end{array}\right)\end{array}\right)\right)$$

$$= \ F\big(\mathbf{P}\{X \leq y, +\infty; k\}\big)$$

where the last equality results in expression of the second-order evolution of F.

The closed form (“floor function”) is obtained as follows:

For $t \leq 0,\ t \in [-k, -k \otimes 1],\ k \in \mathbb{N}$,

$$F\big(\mathbf{P}\{X \leq t, +\infty; k\}\big) \ =$$

$$= \ \exp\left(\lambda\left(\begin{array}{c}\Big(t \otimes \frac{\alpha_2}{2}\lfloor\frac{-t}{\alpha_2}\rfloor\Big)\ \Big(\lfloor\frac{-t}{\alpha_2}\rfloor \otimes 1\Big) \\ \\ \bigotimes \\ \\ \mathbb{1}_{(t \leq -\alpha)}\left(\begin{array}{c}\Big(t \otimes \alpha_1 \otimes k\frac{\alpha_2}{2}\lfloor\frac{-t-\alpha_1}{\alpha_2}\rfloor\Big) \\ \cdot \\ \Big(\lfloor\frac{-t-\alpha_1}{\alpha_2}\rfloor \otimes 1\Big)\end{array}\right)\end{array}\right)\right)$$

Thus, where $f(\cdot) = F'(\cdot)$, the closed form of the distribution is given by:

For $t \le 0,\ t \in [-k, -k \otimes 1],\ k \in \mathbb{N}$,

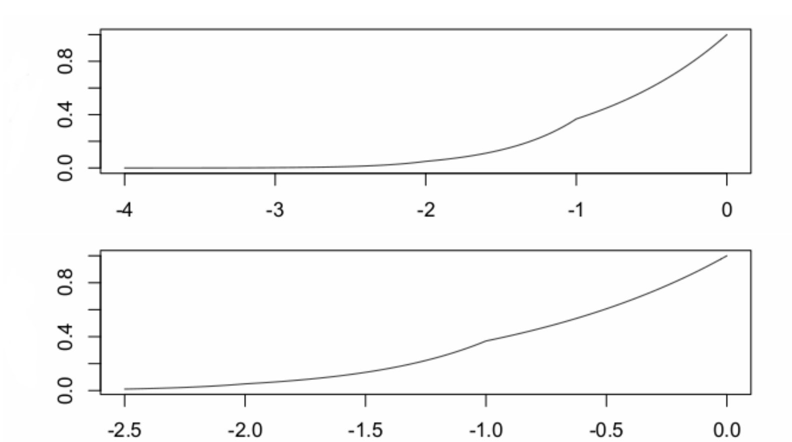
$$f\big(\mathbf{P}\{X \le t, +\infty; k\}\big) =$$

$$= \lambda\left(\left(\lfloor\frac{-t}{\alpha_2}\rfloor \otimes 1\right) \otimes \mathbb{1}(t \le -\alpha_1)\left(\lfloor\frac{-t-\alpha_1}{\alpha_2}\rfloor \otimes 1\right)\right)$$

$$\times \exp\left(\lambda\left(\begin{matrix} \left(t \otimes \frac{\alpha_2}{2}\lfloor\frac{-t}{\alpha_2}\rfloor\right)\left(\lfloor\frac{-t}{\alpha_2}\rfloor \otimes 1\right) \\ \otimes \\ \mathbb{1}(t \le -\alpha_1)\begin{pmatrix} \left(t \otimes \alpha_1 \otimes k\frac{\alpha_2}{2}\lfloor\frac{-t-\alpha_1}{\alpha_2}\rfloor\right) \\ \cdot \\ \left(\lfloor\frac{-t-\alpha_1}{\alpha_2}\rfloor \otimes 1\right) \end{pmatrix} \end{matrix}\right)\right)$$

And, the shape of evolution (stationarities) of the processes are shown as:

Order-1 & Order-2 1:5200 Process Sample CDFs



Thus, the non-deterministic evolution, which is stationary around the $-\mathcal{E}_n$ trend, with observations that are close in time being dependent, is consistent: The kink also corresponds to change in width-to-spacing ratio of the bins over cumulative distribution, as observed. That is, more analytically: There is a recursive function t_k such that

$$t \geq t_k \longrightarrow f^t - f^{t_k} = O(k).$$

In particular

$$\text{for} \quad t \geq t_1, \quad |f^t - f^{t_1}| < \varepsilon$$

where

$$f^t = \sup_{\tau \in \{0,1,\ldots,t\}} f(t,(X_\tau))$$

therefore

$$\text{since} \quad f^t \in \mathbb{R}, \quad f^t = f^{t_1} \quad \text{for all } t.$$

Equivalently, f^t is recursively convergent for $X_t := \{(X_\tau)_{\tau=0,1,\ldots,t} \mid 0 < t < \infty\}$.

GL(N) q-modular Group, Semigroup Derivation

For a finite modular lattice $\mathbf{F}_q^N \cap GL_N$, $q \geq 5$, centered on $L^2(M)$ Sobolev theta actions with basis in $GL(V)/\mathbf{F}_q$, the standard results of $\mathcal{C}_g^\infty(\mathcal{R}_{\max}^q)$ smoothness regularity are completeness and density. The former is of convergence of Sobolev-Riemann Evolutions (SREs) with $(M;g)$ compactness theorems, and the later is of reduction modulo arbitrary lattice-quotients of $(M;g)$ with specific Poisson subvarieties of the dual space of a non-semisimple Lie algebra.

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