Knot Logarithmic Vector Field of Grassmannian Moduli Space $G(\cdot\ ,\ \mathfrak{g})$

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Abstract We construct algebro-geometric representations by invariants of knot $\iota^{<\infty} : S^1 \mapsto \mathbb{R}^3$ cohomology $H^*(\overline{\mathcal{M}}_{g;n}, \mathbb{Q})$ sheaves in $\overline{\mathcal{M}}_{g;n}$ “minimal compactifications” with logarithmic vector field of Lie algebra $\mathfrak{g}$ and complements $\{\partial^2 F/\partial t_0^2, \partial^2 K/\partial t_0^2\}$, for $\{F(t_0, t_1, \dots), K(t_0, t_1, \dots)\}$ map generators to the KdV hierarchy $(L_j U = 0)_{j=1,2,\dots}$. Using the notion of moduli space as “space of parameters,” we establish Witten conjecture as a special case of Kontsevich theorem in “projections” of knot theory and embedded graphs, which permit computations in intersection indices $(\cdot, \cdot, \cdot)$ number $\langle \cdots \rangle$ for 4-term moduli-space bundle classes. By Hermitian operators completion with the combinatorial inverses of the strictly-Riemannian fractional exponentials in algebro-geometric form, for Riemann complex sphere $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1$, not apparent in the original Witten formulation, we derive KdV Sato-Grassmannian function $\tau : \overline{\Lambda} \mapsto \mathbb{C}P^1$. And, using the holomorphic isomorphism between the KdV hierarchy and the fundamental group $\left(-z^{m+1}\frac{d}{dz}\right)$ of τ-complements $\{\psi_{n+1}[D_j] = 0\}$ for the discriminant divisors $\{D_j \subset \overline{\mathcal{M}}_{g;n}\}_{j=1,\cdots,n}$, for smooth mapping classes, we prove Witten’s conjecture but generally the Kontsevich theorem and equivalence. From multivariate hypergeometric mapping π characteristic for Bernoulli term-tier $\left\{B_{\lfloor 2g^2/(1+g)\rfloor}, \widetilde{B}_{\sqrt{-1}\,t}\right\}$ operators of Möbius inverses, we formulate general enumerative kernel $\mathcal{Z}_{g;(\cdot)}$ for (finite) parameterization classes of all *a priori* unknown empirical structures of the succeeding cohomology rings in $\mathbb{C}$. In particular, we prove the formulated kernel is also a solution to the Hurwitz problem $h_{g;(\cdot)}$ known in rather explicit form of pairs (X, f) parameterization, where X is a curve and f is meromorphic function on X.

Keywords: Knot, KdV Grassmannian, logarithmic vector field

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Introduction

At the beginning of the 1990's, V. Vassiliev introduced a new class of invariants, of order not greater than $n < \infty$ double points for every knot, that is, the smooth mapping $\iota^{<\infty} : S^1 \mapsto \mathbb{R}^3$ representing the canonical embedding of an oriented circle S^1 into $\mathbb{R}^3$. Thus, by restriction $\mathbb{R}^3 \backslash \mathbb{R}^3_{\iota<\infty}$, the locally compact fibration of rational cohomology $H^*(\overline{\mathcal{M}}_{g;n}, \mathbb{Q})$ rings of invariants, we have simplicial families explicitly canonical in the sense of "minimal compactifications" for moduli spaces $\overline{\mathcal{M}}_{g;n}$ of a 4-term structure, where the invariants are functions of mapping $\hat{b}_m : \mathcal{A}_n \to \mathcal{B}_m$ which generates quotient module $\mathcal{M}_n = \mathcal{C}_n/\mathcal{C}_n^{(4)} \cong \mathcal{A}_n/\mathcal{A}_n^{(4)}$, with isomorphism $\mathcal{A}_n/\mathcal{A}_n^{(4)} \cong \mathcal{C}_n/\mathcal{C}_n^{(4)}$ taking class $[a] \in \mathcal{A}_n/\mathcal{A}_n^{(4)}$ of arc diagram a to class $[c] \in \mathcal{C}_n/\mathcal{C}_n^{(4)}$ of chord diagram c represented by the algebra of arc diagram a of order n (or n arcs), such that the values in the ℓ-algebra $\bar{\ell}$ form ℓ-module $\mathcal{V}_n$ and ℓ-module $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2 \cup \ldots$ of finite order invariants endowed with filtration $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}$. By a parameterization of intersections, for all algebraic curves with a completion in the ψ-classes of all genus-g modular compactifications, the string equation is derived as:

$$\left\langle \tau_0 \prod_{i=m_1}^{m_n} \tau_i \right\rangle = \left\langle \prod_{i=m_1-1}^{m_n} \tau_i \right\rangle + \cdots + \left\langle \prod_{i=m_1}^{m_n-1} \tau_i \right\rangle \Bigg| \left(\sum_{m_1,\ldots,m_{n+1}} 1 \right) = (n+1) - 3\,.$$

In genus zero, in particular, for all rational curves' intersection indices, this implies $\left\langle \prod_{i=m_1}^{m_n} \tau_i \right\rangle = \begin{pmatrix} n-3 \\ \prod_{i=m_1}^{m_n} i \end{pmatrix} = \frac{(n-3)!}{\prod_{i=m_1}^{m_n} i!} \Bigg| \left(\sum_{m_1,\ldots,m_n} 1 \right) = n-3$. Deriving the dilaton equation as $\left\langle \tau_1 \prod_{i=m_1}^{m_n} \tau_i \right\rangle = (2g-2+n) \left\langle \prod_{i=m_1}^{m_n} \tau_i \right\rangle$ then, explicit representation for all elliptic curves' intersection indices (of no specific parameter "genus", since the formulation is determined by the set of indices $m_1, \ldots, m_n$) is given by:

$$\left\langle \prod_{i=m_1}^{m_n} \tau_i \right\rangle = \frac{1}{24} \begin{pmatrix} n \\ \prod_{i=m_1}^{m_n} i \end{pmatrix} \left(1 - \sum_{i=2}^{n} \frac{(i-2)!(n-i)!}{n!}\, e_i(m_1,\ldots,m_n) \right) \quad \Bigg| \quad \left(\sum_{m_1,\ldots,m_n} 1 \right) = n.$$

Here, $e_k(m_1,\ldots,m_n) := \sum_{i_1<\ldots<i_k} \prod_{\eta=1}^{k} m_{i_\eta}$ is the kth elementary symmetric function; and, $(2g-2+n)$, which is the number of zeros of meromorphic 1-form on genus-g curve with poles of order one at n marked points, is the

integral of ψ_{n+1} over all fibers of $\pi : \overline{\mathcal{M}}_{g;n+1} \to \overline{\mathcal{M}}_{g;n}$ such that the divisors $\{D_j \subset \overline{\mathcal{M}}_{g;n} \mid \psi_{n+1}[D_j] = 0\}_{j=1,\cdots,n}$ consist of curves containing a smooth rational irreducible component with only marked points $(x_j\,,\,x_{n+1})$ on π intersecting other components at a single point, within the whole kernel $\mathcal{Z}_{g;k}$. Moreover, first Chern classes $\{\psi'_j,\,\psi_j\}$ of holomorphic line bundles of the 4-term moduli spaces $\overline{\mathcal{M}}_{g;n}$ and $\overline{\mathcal{M}}_{g;n+1}$ are generated over Hodge integrals by $\left\langle \tau_1 \prod_{i=m_1}^{m_n} \tau_i \right\rangle = \int_{\overline{\mathcal{M}}_{g;n+1}} \Big(\prod_{i=1}^{n} (\pi^*(\psi'_i))^{m_i} \Big) \psi_{n+1}$.

Unifying all intersection numbers of ψ-classes into exponential generating function $F(t_0, t_1, \dots) := \left\langle \exp\left(\sum_i t_i \tau_i \right) \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(m_1,\dots,m_n)} \left\langle \prod_{i=m_1}^{m_n} \tau_i \right\rangle \prod_{i=m_1}^{m_n} t_i = = \sum_{(l_0,\dots,l_s)} \left\langle \prod_{i=0}^{s} \tau_i^{l_i} \right\rangle \prod_{i=0}^{s} \frac{t_i^{l_i}}{l_i!}$ implies F in the variables t_i is a τ-function for the Korteweg-de Vries (KdV) hierarchy $(L_j U = 0)_{j=1,2,\dots}$, that is, the (not necessarily homogeneous) system of linear equations for fixed differential operators L_j of order $j = 1, 2, \cdots$, on the universal one-matrix model expressed in the power series $U(t_0, t_1, t_2, \dots) := \sum_{g=0}^{\infty} \sum_{\tau_i} \prod_{i=1}^{\infty} \frac{t_i^{\tau_i}}{\tau_i!} e_g(\tau_1, \tau_2, \dots)$ of infinitely many variables $t_0 = y, t_1, t_2, \dots$ such that $V(t_0, t_1, \dots) := \partial^2 F / \partial t_0^2$ is a solution to the KdV equation. In particular, string and dilaton equations are the PDEs $L_{-1}V = 0$ and $L_0 V = 0$, respectively. And, generating function F satisfies the system of equations $L_m(F) = 0$, for $m \geq -1$, where L_m belongs to the KdV hierarchy's operators with commutation relations $\big[L_m\,, L_n\big] := (n-m)\, L_{n+m}$ such that the hierarchy spans the Lie algebra (generated by L_{-1} and L_2) which is isomorphic to the Lie algebra of polynomial vector fields on the line under the isomorphism $L_m \mapsto -z^{m+1} \frac{d}{dz}$.

Thus, by the latter isomorphism, a unique extension of the representation of Lie algebra of polynomial vector fields generalizes to algebra of differential operators in t_k starting with L_{-1} and L_0 under homogeneity conditions. This implies the algebro-geometric proof of Witten's conjecture, but more generally, of Kontsevich's theorem, given that the intermediary between the constructed intersection model (of Vassiliev-type knot-invariant algebraic curves parameterization) and the one-matrix model is the Kontsevich's theorem, with an exciting relation of being equivalent to both the one-matrix model and the intersection model; in particular, with positive definite diagonal $N \times N$ matrix Λ of entries $\Lambda_1, \dots, \Lambda_N$, a new measure $d\mu_\Lambda(H) := C_{\Lambda,N}\, e^{-\frac{1}{2}\,\mathrm{tr}\, H^2\Lambda} dv(H)$ with volume form $dv(H)$ on Hermitian matrix-space $\Big\{H := (h_{kl}) \;\;|\;\; h_{lk} = \bar{h}_{kl}, \;\; h_{kl} = x_{kl} + iy_{kl}\Big\}$ is constructed, with mean $\overline{h_{ij}h_{kl}}$ given by $\langle h_{ij}\, h_{kl}\rangle$ for degree-2 monomial $h_{ij}h_{kl}$, and $C_{\Lambda,N}$ chosen to guarantee $\displaystyle\int_{\mathcal{H}_N} d\mu_\Lambda(H) = 1$. Thus, by the characteristic $d\mu_\Lambda(H)$, expanding $\log\displaystyle\int_{\mathcal{H}_N} e^{\frac{i}{6}\,\mathrm{tr}\, H^3} d\mu_\Lambda(H) \;\; = \;\; \log\int_{\mathcal{H}_N}\Big(1 - \frac{1}{2!}\frac{1}{6^2}(\mathrm{tr}\, H^3)^2 + \frac{1}{4!}\frac{1}{6^4}(\mathrm{tr}\, H^3)^4 \;-\; \cdots$ $\cdots\Big) d\mu_\Lambda(H) = \log\Big(1 + \frac{1}{3!}t_0^3 + \frac{1}{24}t_1 + \frac{25}{144}t_0^3 t_1 + \frac{1}{124416}t_0^6 + \cdots\Big) := K(t_0, t_1, \cdots)$, such that the $K(t_0, t_1, \ldots)$-integral is a τ-function for the KdV hierarchy, implies the second derivative $\partial^2 K/\partial t_0^2$ is a solution to the KdV equation, treating the function K as a matrix Airy function. Hence, on the one hand, the coefficient of $\dfrac{\prod_{i=0}^{s}\tau_i^{l_i}}{\prod_{i=0}^{s} l_i!}$ coincides with the intersection number $\Big\langle\displaystyle\prod_{i=0}^{s}\tau_i^{l_i}\Big\rangle$ in the expansion of $K(t_0, t_1, \ldots)$-integral in the variables t_i. On the other hand, $V(t_0, t_1, \ldots) \;\; := \;\; \partial^2 F/\partial t_0^2$ coincides with universal one-matrix partition function for $U(t_0, t_1, t_2, \ldots)$ in the expansion of $K(t_0, t_1, \cdots)$-integral, where $e_g(\tau_1, \tau_2, \ldots)$ represents leading-term coefficients $e_0, e_1, \ldots$ in the expansion of singular part of the functions $e_g(\cdot)$ around the critical point $t = t_c$ with vanishing imaginary parts (since the number of ways to glue an odd number of 3-stars is zero).

Hence, by the projection $\pi : \mathcal{M}_{g;n}^c \cong \mathcal{M}_{g;n} \times \mathbb{R}_+^n \to \mathbb{R}_+^n$ which takes a marked graph with a metric to n-tuple of perimeters $p_1, \ldots, p_n$ of marked-points, where e' and e'' run over all pairs of distinct edges of every ith face, with e' preceding e'' in fixed order of a chosen starting vertex, the class ψ_i is represented by the real 2-forms $\omega_i \;:=\; \sum d(l_{e'}/p_i) \wedge d(l_{e''}/p_i)$ defined only on open strata of the combinatorial $\mathcal{M}_{g;n}^c$ to the second factor. Fixing smooth curve $(X; x_1, \ldots, x_n)$, the vertical trajectories induced by canonical Jenkins-Strebel quadratic differential for n-tuple $p_1, \ldots, p_n$ through x_i identifies perimeter of the ith face of corresponding embedded graph with "spherized" cotangent line L_i considered as a real plane at the ith point. Hence, the fiber punctured at origin is projected to unit circle along half-lines through the origin; and intersection numbers are represented in terms of integrals of very explicit differential forms: $\Big\langle\displaystyle\prod_{i=m_1}^{m_n}\tau_i\Big\rangle = \int_{\pi^{-1}(\bar{p})}\prod_{i=1}^{n}\omega_i^{m_i}$ for any generic point $\bar{p} \in \mathbb{R}_+^n$ with volume form $\mathrm{Vol}(\lambda_1, \ldots, \lambda_n) = \dfrac{1}{d!}\Omega^d \times \displaystyle\prod_{i=1}^{n} e^{-\lambda_i p_i} dp_i$ on open strata of $\mathcal{M}_{g;n}^c$ such that $d = 3g - 3 + n$ is complex dimension of $\mathcal{M}_{g;n}$,

$\Omega := \sum_{i=1}^{n} p_i^2 \omega_i$, and λ_i are real positive parameters. Thus, the volume of $\mathcal{M}_{g;n}^c$ with respect to the volume form is realized in two ways. First, directly under projection unto $\mathbb{R}_+^n$:

$$\begin{aligned}
\int_{\mathcal{M}_{g;n}^c} \mathsf{Vol}(\lambda_1, \ldots, \lambda_n) &= \frac{1}{d!} \int_{\mathbb{R}_+^n} \Big(\int_{\pi^{-1}(\bar{p})} \Omega^d \Big) e^{-\sum \lambda_i p_i} \bigwedge_{i=1}^{n} dp_i \\
&= \sum_{\left(\left(\sum_{i=m_1}^{m_n} i\right) = d\right)} \frac{\left\langle \prod_{i=m_1}^{m_n} \tau_i \right\rangle}{\prod_{i=m_1}^{m_n} i!} \prod_{i=1} \int_0^\infty p_i^{2m_i} e^{-\lambda_i p_i} dp_i \\
&= \sum_{\left(\left(\sum_{i=m_1}^{m_n} i\right) = d\right)} \frac{\left\langle \prod_{i=m_1}^{m_n} \tau_i \right\rangle}{\prod_{i=m_1}^{m_n} i!} \prod_{i=1}^{n} \frac{(2m_i)!}{m_i!} \lambda_i^{-(2m_i+1)} \\
&= 2^d \sum_{\left(\left(\sum_{i=m_1}^{m_n} i\right) = d\right)} \frac{\left\langle \prod_{i=m_1}^{m_n} \tau_i \right\rangle}{\prod_{i=m_1}^{m_n} i!} \prod_{i=1}^{n} \frac{(2m_i-1)!}{\lambda_i^{(2m_i+1)}} .
\end{aligned}$$

Secondly, summing up volumes of all open cells in $\mathcal{M}_{g;n}^c$, where an open cell corresponds to a 3-valent embedded graph Γ, with lengths $l_1, \ldots, l_{|E(\Gamma)|}$ of edges forming a set of coordinates on the cell, the volume form $\mathsf{Vol}(\lambda_1, \ldots, \lambda_n)$ is specified by $\mathsf{Vol}_\Gamma(\lambda_1, \ldots, \lambda_n) := 2^{d+|E(\Gamma)|-|V(\Gamma)|} e^{-\sum_j l_j \bar{\lambda}_j} dl_1 \wedge \cdots \wedge dl_{|E(\Gamma)|}$ and independent of the chosen cell (show ♡), where j runs over the set of all edges of Γ, and $\bar{\lambda}_j$ is sum $\bar{\lambda}_j := \lambda_- + \lambda_+$ of the two $\lambda' s$ corresponding to the two faces of Γ adjacent to the jth edge. In particular, for coinciding two faces neighboring to an edge: $\lambda_- = \lambda_+$ and $\mathsf{Vol}_\Gamma(\lambda_1, \ldots, \lambda_n) = \prod_{j=1}^{|E(\Gamma)|} \frac{1}{\bar{\lambda}_j}$. And, since the contribution of a marked embedded graph to total volume is proportional to the inverse-cardinality of the automorphism group of the graph, summing over all 3-valent marked genus-g embedded graphs with n marked faces and multiplying by 2^{-d} gives the main combinatorial identity (in variables λ_i between two rational functions):

$$\sum_{\left(\left(\sum_{i=m_1}^{m_n} i\right) = d\right)} \left\langle \prod_{i=m_1}^{m_n} \tau_i \right\rangle \prod_{i=1}^{n} \frac{(2m_i-1)!!}{\lambda_i^{(2m_i+1)}} \quad = \quad \sum_\Gamma \frac{2^{-|V(\Gamma)|}}{|\mathsf{Aut}(\Gamma)|} \prod_{j=1}^{|E(\Gamma)|} \frac{2}{\tilde{\Lambda}_j} .$$

In addition, summing the resulting identities over all arbitrary substitution of the form $\lambda_i = \lambda_{k_i}$, $1 \leq k_i \leq N$, gives:

$$\sum_{\left(\left(\sum\limits_{i=m_1}^{m_n} i\right)=d\right)} \left\langle \prod_{i=m_1}^{m_n} \tau_i \right\rangle \prod_{i=1}^{n} (2m_i - 1)!! \operatorname{tr} \Lambda^{(-2m_i-1)} = \sum_{\Gamma} \frac{2^{-|V(\Gamma)|}}{|\mathrm{Aut}(\Gamma)|} \prod_{j=1}^{|E(\Gamma)|} \frac{2}{\tilde{\Lambda}_j}$$

where $\tilde{\Lambda} := \Lambda_- + \Lambda_+$ and the sum on the right hand side is taken over all possible ways to color the faces of Γ in the N colors $\Lambda_1, \ldots, \Lambda_N$ which are the entries of the $N \times N$ positive definite Λ. That is, the right hand side coincides with the matrix integral expansion of the Kontsevich model; namely, generating function K of the Kontsevich matrix integral coincides with the generating function F of the intersection model; and, this completes proof of first part of Kontsevich theorem.

Now, treating the integral of Kontsevich as a τ-function for the KdV-hierarchy: In other words, obeying the KdV equation, an "asymptotic behavior" is observed from matrix Airy function $A(Y) := \int_{\mathcal{H}_N} e^{i(\frac{1}{3} \operatorname{tr} H^3 - HY)} d\mu(H)$ of positive diagonal matrix Y satisfying matrix Airy equation $\Delta A(Y) + \operatorname{tr} Y \cdot A(Y) = 0$, where Δ is Laplace operator. In particular, the asymptotic is similar to that of classical Airy function $a(y) := \int_{-\infty}^{\infty} e^{i(\frac{1}{3}x^3 - yx)} dx$ of unique bounded solution, up to scalar factor, for linear differential equation $a''(y) + y\, a(y) = 0$, as $y \to \infty$ with stationary phase:

$$a(y) \sim e^{-\frac{2i}{3} y^{3/2}} \int_{U(y^{1/2})} e^{i(\frac{1}{3}x^3 + y^{1/2}x^2)} dx + e^{\frac{2i}{3} y^{3/2}} \int_{U(y^{1/2})} e^{i(\frac{1}{3}x^3 - y^{1/2}x^2)} dx$$

in arbitrary neighborhoods of the points $\pm y^{1/2}$.

Thus, similar to the 1-dimensional Airy function, the matrix Airy function admits an asymptotic expansion as a sum of 2^N expressions of the form:

$$e^{-i\,\frac{2}{3} \operatorname{tr} Y^{3/2}} \int e^{-i \operatorname{tr} (\frac{1}{3}H^3 - H^2 Y^{1/2})} d\mu(H) = e^{-i\,\frac{2}{3} \operatorname{tr} Y^{3/2}} \int e^{-i \operatorname{tr} \frac{1}{3} H^3} d\mu_{Y^{1/2}}(H)$$

such that the sum is taken over all 2^N quadratic roots $Y^{1/2}$ of the matrix Y with the integral taken over a neighborhood of the origin in $\mathcal{H}_N$. And, as $Y \to \infty$, by extending integration to the entire space of $\mathcal{H}_N$, the integral becomes the Kontsevich model for $\Lambda := Y^{1/2}$ whose asymptotic expansion is already known. Hence, with the Vandermonde determinant Δ:

$$A(Y) := c_N \Delta(Y_i)^{-1} \int_{\mathbb{R}^N} \prod_{i=1}^{n} \Delta(X_i) e^{i(\frac{1}{3}X_i^3 - X_i Y_i)} dX_i = c_N \frac{\det(a^{(j-1)}(Y_i))}{\det(Y_i^{j-1})} .$$

By identity $\int e^{i(x^3/3-xy)}x^{j-1}dx = (ia(y))^{(j-1)}$, derivatives of the Airy function admit natural asymptotic expansions $a^{(j-1)}(y) \sim \sum_{y^{1/2}} \text{const} \cdot y^{-3/4}e^{-\frac{2i}{3}y^{3/2}} \cdot f_j(y^{-1/2})$ for the Laurent series $f_j(z) := z^{-j} + \cdots \in \mathbb{Q}((z))$. Hence, substituting the asymptotic expression into the asymptotic-generating matrix Airy function implies $A(Y) := \sum_{Y^{1/2}} \text{const} \times e^{-\frac{2i}{3}trY^{3/2}} \prod_{i=1}^{N} Y_i^{-3/4} \cdot \frac{\det(f_j(Y^{-1/2}))}{\det(Y_i^{j-1})}$, which relates the matrix Airy function to the τ-function corresponding to the conventional subspace $\langle f_1, f_2, \ldots \rangle \subset \mathbb{C}((Z^{-1}))$ of the infinite dimensional space of Laurent series in z^{-1}, that is, the form of $f(z^{-1}) := \sum_{i=-\infty}^{\infty} a_i z^{-i}$. And, this completes the proof of Witten's conjecture, since the former is invariant under multiplication by z^{-2} and further satisfies the KdV hierarchy's Sato-Grassmannian:

$$\begin{cases} \frac{\partial S}{\partial T_{2k+1}} = [S_+^{(2k+1)/2}, S] \\ S(y, T_1, T_3, T_5, \ldots) = \frac{\partial^2}{\partial y^2} + 2\frac{\partial^2}{\partial t_1^2} \log \tau_W(y + T_1, T_3, T_5, \ldots) \end{cases}$$

with determinant $\tau_W(T_1, T_3, T_5, \ldots)$.

In principle, by Kontsevich theorem, the $\left\langle \cdots \prod \cdots \right\rangle$ in form of Hodge integrals

$$\left\{ \int_{\overline{\mathcal{M}}_{g;n}} \Big(\prod_{i=1}^{n} \psi_i^{m_i}\Big)\lambda_g = \begin{pmatrix} n \\ \prod_{i=m_1}^{m_n} i \end{pmatrix} b_g \;\middle|\; \Big(\sum_{m_1,\ldots,m_n} 1\Big) = n\,, \quad b_g = \int_{\overline{\mathcal{M}}_{g;n}} \psi_1^{2g-2}\lambda_j \right\}$$

$\forall j > 0$, are a constant depending only on the genus, with respect to monomials containing only one λ-class, including the highest degree λ_g, such that $\left(1 + \sum_{g=1}^{\infty} b_g t^{2g}\right) = \frac{t/2}{\sin(t/2)} \;\Big|\; b_g = \frac{2^{2g-1}-1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}$, where each B_{2g} is the $(2g)$th Bernoulli number, and the b's are given by $b_1 = \frac{1}{24}$, $b_2 = \frac{7}{5760}$, $b_3 = \frac{31}{967680}, \cdots$, from matching coefficients of the exponential generating function:

$$\frac{t}{e^t - 1} = \sum_{g=0}^{\infty} (-1)^g \frac{B_{\lfloor 2g^2/(1+g) \rfloor}}{\lfloor 2g^2/(1+g) \rfloor !} t^{\lfloor 2g^2/(1+g) \rfloor} = 1 - \frac{1}{2}t + \frac{B_2}{2!}t^2 + \frac{B_4}{4!}t^4 + \cdots$$

where $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$, $B_{10} = \frac{5}{66}, \cdots$.

Hence, the whole (enumerable) kernel $\mathcal{Z}_{g;k}$, which is understood as the closure of tuples $\{(p_i)_{i=1}^n, n < \infty\}$ of principal parts (at genus-g marked points $x_1, \ldots, x_n$), is associated with a meromorphic class on not only smooth curves but also on all stable curves; in other words, the complete Hurwitz space $\overline{\mathcal{H}}_{g;k}$ can be different from the whole kernel $\mathcal{Z}_{g;k}$. That is, not every meromorphic function on a singular curve is the limit of a family of meromorphic functions on smooth curves; however, points of the complete Hurwitz spaces $\overline{\mathcal{H}}_{g;k}$, and of the whole kernel space $\mathcal{Z}_{g;k}$, can be enumerated as stable mappings from genus-g curves to $\mathbb{C}P^1$ both in the sense of Kontsevich and in the specific case of mappings of curves to $(\mathbb{C}P^1, \infty)$, where every holomorphic mapping $f : (X; x_1, \ldots, x_n) \to (\mathbb{C}P^1, \infty)$ of a nodal genus-g curve X to the projective line, taking marked nonsingular points $x_1, \ldots, x_n$ (and only these points) to infinity, is deemed stable if its automorphism group is finite. Moreover, each irreducible component X taken by f to a single point possesses at least three singular points if its genus is zero – at least one singular point if its genus is one.

Thus, under ramified covering of cohomology classes, a stronger more general theorem follows: To wit, for every genus-g with n marked points, there are $k < n$ irreducible components in the kernel $\mathcal{Z}_{g;(\cdot)}$ coinciding with the complete Hurwitz space $\overline{\mathcal{H}}_{g;(\cdot)}$ and $n-k$ consisting of functions over reducible curves that are constant on the elliptical component, while their intersection consists of functions on reducible curves: such that their value at the double point coincides with a critical value of their restriction to the rational component. As a corollary, enumeration of genus-1 Hurwitz numbers, alongside the derived string and dilaton equations, is explicitly given by:

$$h_{1;k} \quad = \quad \frac{(k+n)!}{24\,|\mathsf{Aut}(k_1,\ldots,k_n)|} \prod_{i=1}^{n} \frac{k_i^{k_i}}{k_i!} \left(k^n - \sum_{i=2}^{n} (i-2)!\, e_i k^{n-i} - k^{n-1} \right)$$

where $e_i := e_i(k_1, \ldots, k_n)$ is the ith elementary symmetric function in $k_1, \ldots, k_n$, such that $k := \sum_{i=1}^{n} k_i = e_1$, and k^{n-1} is total number of the λ_1-containing integrals. In addition, the invertible local mapping (of the set of all non-linear operators) by $\widetilde{B}_{\sqrt{-1}\,t} := 1 - t + \frac{\widetilde{B}_2}{2!}t^2 - \frac{\widetilde{B}_3}{3!}t^3 + \frac{\widetilde{B}_4}{4!}t^4 + \ldots$ admits the conformal weight series:

$$\left\{ 1 \quad + \lim_{g\to\infty} \sum_{h=1}^{\lfloor (g^2+g)/(g+1) \rfloor} b_{h/2} t^h \quad = \quad \frac{t/2}{\sin(t/2)} \;\middle|\; b_{h/2} \quad := \quad \frac{2^{h-1}-1}{2^{h-1}} \frac{|\widetilde{B}_g|}{g!} \right\}$$

for all $\left(\sum_g (-1)^g \frac{\widetilde{B}_{\lfloor (g^2+g)/(g+1) \rfloor}}{\lfloor (g^2+g)/(g+1) \rfloor\,!}\, t^{\lfloor (g^2+g)/(g+1) \rfloor} \right) < \infty$ such that B_{2g}, which is a Bernoulli number, is the gth inverse Möbius transform of the series $(\widetilde{B}_{\sqrt{-1}\,t} \mid \forall t)$.

Universal Matrix Model

Definition