Lie Algebra Weight System

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Abstract

We prove construction of weight system ω_s with respect to Lie algebras $\mathfrak{sl}_2$, $\mathfrak{gl}_N$ by 4-term relations in Vasiliev knot invariants; applications include Knizhnik – Zamolodchikov – Kontsevich (KZK) and the Knizhnik – Zamolodchikov – Bernard (KZB) weight systems for knot invariant: a constant characteristic of an arbitrary set of knots which is independent of representation (although the set is not without a well defined representation) with respect to ambient isotopy.

1 Lie alg wt sys for the algebra $\mathcal{A}^{fr}$

1.1 Universal Lie algebra weight systems

Following Kontsevich's: Let $\mathfrak{g}$ be a metrized Lie algebra over $\mathbb{R}$ or $\mathbb{C}$, that is, a Lie algebra with an ad-invariant non-degenerate bilinear form $\langle\cdot,\cdot\rangle$:

$$\beta \quad : \quad \mathfrak{g}\times\mathfrak{g}\longrightarrow\mathbb{R}$$

$$\beta([X,Y],Z) \;=\; -\beta(Y,[X,Z]), \quad \forall\, X,Y,Z\in\mathfrak{g}$$

$\omega_s =$ trace of operator for finite representation of $\mathfrak{g}$

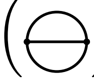
$\omega_s \in ZU(\mathfrak{g}) =$ center of universal enveloping algebra $U(\mathfrak{g})$.

Remark. $\omega_s =$ symbols of quantum group invariants (since quantum invariant is polynomial in q and q^{-1}, and substituting $q = e^h$, coefficient of h^n can be considered in Taylor expansion of the quantum invariant. Another way of constructing weight systems, also due to Kontsevich is using *marked surfaces*.

1.2 Vasilievs Derivation of Lie Algebra Wt Sys

Derivation I

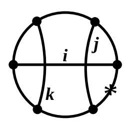
$$\omega_s = \varphi_{\mathfrak{g}}\Big(\ \Big)$$



$$= \sum_{i=1}^{m} e_i e_i^* \qquad \text{i.e., the Quadratic Casimir}$$

Derivation II

$$\omega_s = \varphi_{\mathfrak{g}}\Big(\ \Big)$$



$$= \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{m} e_i e_j e_k e_i^* e_k^* e_j^*$$

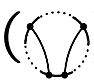
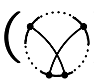
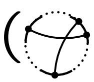
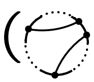
Theorem I

1. $\varphi_{\mathfrak{g}}(D)$ belongs to the center $ZU(\mathfrak{g})$ ad-invariant subspace

$$U(\mathfrak{g})^{\mathfrak{g}} = \{x \in U(\mathfrak{g}) \mid xy = yx,\ \forall\, y \in \mathfrak{g}\}$$

of the universal enveloping algebra $U(\mathfrak{g})$.

2. The function $f : D \longmapsto \varphi_{\mathfrak{g}}(D)$ satisfies 4-term relations

$f($  $) - f($  $) + f($  $) - f($  $) = 0.$

3. The resulting map $\varphi_{\mathfrak{g}} : \mathcal{A}^{fr} \to ZU(\mathfrak{g})$ is homomorphism of algebras.

Proof

1. $\varphi_{\mathfrak{g}}(D)$ commutes with any basis element e_r. So, choose $e_i^* = e_i$ for all i, and expand commutator of e_r and $\varphi_{\mathfrak{g}}(D)$ into sum $2n$ of expressions, similar to $\varphi_{\mathfrak{g}}(D)$, only with one of e_i's replaced by its commutator with e_r. Concretely,

$$
\begin{aligned}
&[e_r, \sum_{ij} e_i e_j e_i e_j] \\
=&\sum_{ij}[e_r, e_i]e_j e_i e_j + \sum_{ij} e_i[e_r, e_j]e_i e_j + \sum_{ij} e_i e_j[e_r, e_i]e_j + \sum_{ij} e_i e_j e_i[e_r, e_j] \\
=&\sum_{ijk} c_{rik} e_k e_j e_i e_j + \sum_{ijk} c_{rjk} e_i e_k e_i e_j + \sum_{ijk} c_{rik} e_i e_j e_k e_j + \sum_{ijk} c_{rjk} e_i e_j e_i e_k \\
=&\sum_{ijk} c_{rik} e_k e_j e_i e_j + \sum_{ijk} c_{rjk} e_i e_k e_i e_j + \sum_{ijk} c_{rki} e_k e_j e_i e_j + \sum_{ijk} c_{rkj} e_i e_k e_i e_j.
\end{aligned}
$$

First sum cancels with the third, and second cancels with the fourth due to antisymmetry property of structure constants c_{ijk}.

2). Still assuming that the basis $\{e_i\}$ is $\langle\cdot,\cdot\rangle$–orthonormal, one of the pairwise differences of the chord diagrams that constitute the 4 term relation is sent by $\varphi_{\mathfrak{g}}$ to

$$\sum c_{ijk} \ldots e_i \ldots e_j \ldots e_k \ldots \ ,$$

while the other goes to

$$\sum c_{ijk} \ldots e_j \ldots e_k \ldots e_i \cdots = \sum c_{kij} \ldots e_i \ldots e_j \ldots e_k \ldots$$

where the equality is due to cyclic symmetry of structure constants c_{ijk} in orthonormal basis.

3). By the indifference wrt basis, we place the base point in the product diagram $D_1 \cdot D_2$ between D_1 and D_2. Therefore, the evident identity $\varphi_{\mathfrak{g}}(D_1 \cdot D_2) = \varphi_{\mathfrak{g}}(D_1)\varphi_{\mathfrak{g}}(D_2)$.

Remark

• If D is a diagram with n chords,

$$\varphi_{\mathfrak{g}}(D) = c^n + \{\text{terms of degree less than } 2n \text{ in } U(\mathfrak{g})\},$$

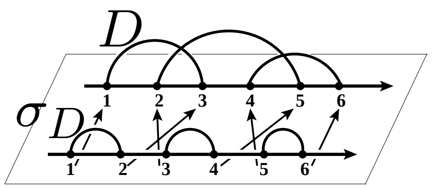
where c is the quadratic Casimir.

• The highest degree term of $\varphi_{\mathfrak{g}}(D)$ does not depend on D, because We can permute the endpoints of chords on the circle without changing the highest term of $\varphi_{\mathfrak{g}}(D)$, and all additional summands arising as commutators have degrees less than $2n$.

• If D is a diagram with n isolated chords, then $\varphi_{\mathfrak{g}}(D) = c^n$, i.e., nth power of diagram with one chord.

• The action of the center $ZU(\mathfrak{g})$ consists in taking the commutator & it's isomorphic to the algebra of polynomials in certain variables $c_1 = c, c_2, \ldots, c_r$, where $r = rank(\mathfrak{g})$.

Derivation III



$\sigma_D = (132546)$

The value of the universal Lie algebra weight system $\varphi_{\mathfrak{g}}(D)$ is then the image of the nth tensor power $\langle \cdot, \cdot \rangle^{\otimes n}$ under the map

$$\mathfrak{g}^{\otimes 2n} \xrightarrow{\sigma_D} \mathfrak{g}^{\otimes 2n} \to U(\mathfrak{g}),$$

where the second map is the natural projection of the tensor algebra on $\mathfrak{g}$ to its universal enveloping algebra.

1.3 Universal $\mathfrak{sl}_2$ weight system

Consider the Lie algebra $\mathfrak{sl}_2$ of 2×2 matrices with zero trace i.e. the three-dimensional Lie algebra spanned by matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with commutators

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H \ .$$

From the symmetric bilinear form property $\langle x, y \rangle = \mathrm{Tr}(xy)$:

$$\langle H, H \rangle = 2,\ \langle H, E \rangle = 0,\ \langle H, F \rangle = 0,$$

$$\langle E, E \rangle = 0,\ \langle E, F \rangle = 1,\ \langle F, F \rangle = 0$$

therefore, ad-invariant and non-degenerate.

Corresponding dual basis is given by

$$H^* = \frac{1}{2}H, \qquad E^* = F, \qquad F^* = E,$$

therefore, the Casimir element is given by $c = \frac{1}{2}HH + EF + FE$.

Remark

• The centre $ZU(\mathfrak{sl}_2)$ is isomorphic to algebra of polynomials in single variable c. The value $\varphi_{\mathfrak{sl}_2}(D)$ is thus a polynomial in c.

• The algebra $\mathfrak{sl}_2$ is simple, hence, any invariant form is equal to $\lambda\langle\cdot,\cdot\rangle$ for some constant λ.

• The corresponding Casimir element c_λ, as an element of the universal enveloping algebra, is related to $c = c_1$ by the formula $c_\lambda = \frac{c}{\lambda}$. Therefore, the weight system

$$\varphi_{\mathfrak{sl}_2}(D) = c^n + a_{n-1}c^{n-1} + a_{n-2}c^{n-2} + \cdots + a_2c^2 + a_1c$$

• and the corresponding weight system to $\lambda\langle\cdot,\cdot\rangle$:

$$\varphi_{\mathfrak{sl}_2,\lambda}(D) = c_\lambda^n + a_{n-1,\lambda}c_\lambda^{n-1} + a_{n-2,\lambda}c_\lambda^{n-2} + \cdots + a_{2,\lambda}c_\lambda^2 + a_{1,\lambda}c_\lambda$$

are related by $\varphi_{\mathfrak{sl}_2,\lambda}(D) = \frac{1}{\lambda^n} \cdot \varphi_{\mathfrak{sl}_2}(D)|_{c=\lambda\cdot c_\lambda}$,

or

$$a_{n-1} = \lambda a_{n-1,\lambda}, a_{n-2} = \lambda^2 a_{n-2,\lambda}, \ldots, a_2 = \lambda^{n-2}a_{2,\lambda}, a_1 = \lambda^{n-1}a_{1,\lambda}.$$

Theorem II

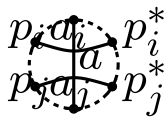
Let $\varphi_{\mathfrak{sl}_2} =$ weight system for $\mathfrak{sl}_2$, with invariant form $\langle\cdot,\cdot\rangle$. Take a chord diagram D and choose a chord a of D. Then

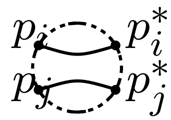
$$\varphi_{\mathfrak{sl}_2}(D) = (c-2k)\varphi_{\mathfrak{sl}_2}(D_a) + 2 \sum_{1\le i<j\le k} \Big(\varphi_{\mathfrak{sl}_2}(D^{\parallel}_{i,j}) - \varphi_{\mathfrak{sl}_2}(D^{\times}_{i,j})\Big),$$

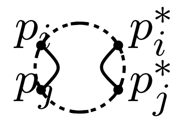
where:

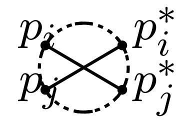
- $= k$ number of chords that intersect the chord a;
- $D_a =$ chord diagram obtained from D by deleting chord a;
- $D^{\parallel}_{i,j}$ and $D^{\times}_{i,j}$ are the chord diagrams obtained from D_a by:

Drawing diagram D so that chord a is vertical; considering arbitrary pair of chords a_i and a_j different from a and such that each intersects a; and, denoting by p_i and p_j the endpoints of a_i and a_j lieing to left of a and by p_i^*, p_j^* the endpoints of a_i and a_j lieing to the right; such that there are three ways to connect the four points p_i, p_i^*, p_j, p_j^* by two chords. D_a is diagram where these two chords are $(p_i, p_i^*), (p_j, p_j^*)$, the diagram $D^{\shortparallel}_{i,j}$ has the chords $(p_i, p_j), (p_i^*, p_j^*)$ and $D^{\times}_{i,j}$ has the chords $(p_i, p_j^*), (p_i^*, p_j)$. All other chords are the same in all the diagrams:

$D =$ 

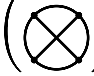
$D_a =$ 

$D^{\shortparallel}_{i,j} =$ 

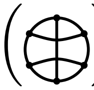
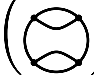
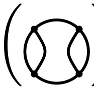
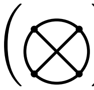
$D^{\times}_{i,j} =$  .

Remark. The theorem allows recursive computation of $\varphi_{\mathfrak{sl}_2}(D)$, as each of diagrams $D_a, D^{\shortparallel}_{i,j}$, $D^{\times}_{i,j}$ has one chord less than D.

Derivation IV

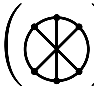
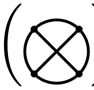
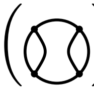
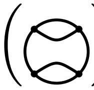
$\varphi_{\mathfrak{sl}_2}\Big($  $\Big) = (c-2)c$. In this case, $k = 1$ and the sum in the right hand side is zero, since no pairs (i, j).

Derivation V

$\varphi_{\mathfrak{sl}_2}\Big($  $\Big) = (c-4)\varphi_{\mathfrak{sl}_2}\Big($  $\Big) + 2\varphi_{\mathfrak{sl}_2}\Big($  $\Big) - 2\varphi_{\mathfrak{sl}_2}\Big($  $\Big)$

$$= (c-4)c^2 + 2c^2 - 2(c-2)c \quad = \quad (c-2)^2c.$$

Derivation VI

$\varphi_{\mathfrak{sl}_2}\Big($  $\Big) = (c-4)\varphi_{\mathfrak{sl}_2}\Big($  $\Big) + 2\varphi_{\mathfrak{sl}_2}\Big($  $\Big) - 2\varphi_{\mathfrak{sl}_2}\Big($  $\Big)$

$$= (c-4)(c-2)c + 2c^2 - 2c^2 \quad = \quad (c-4)(c-2)c.$$

Remark. Choosing invariant form $\lambda\langle\cdot,\cdot\rangle$, we obtain

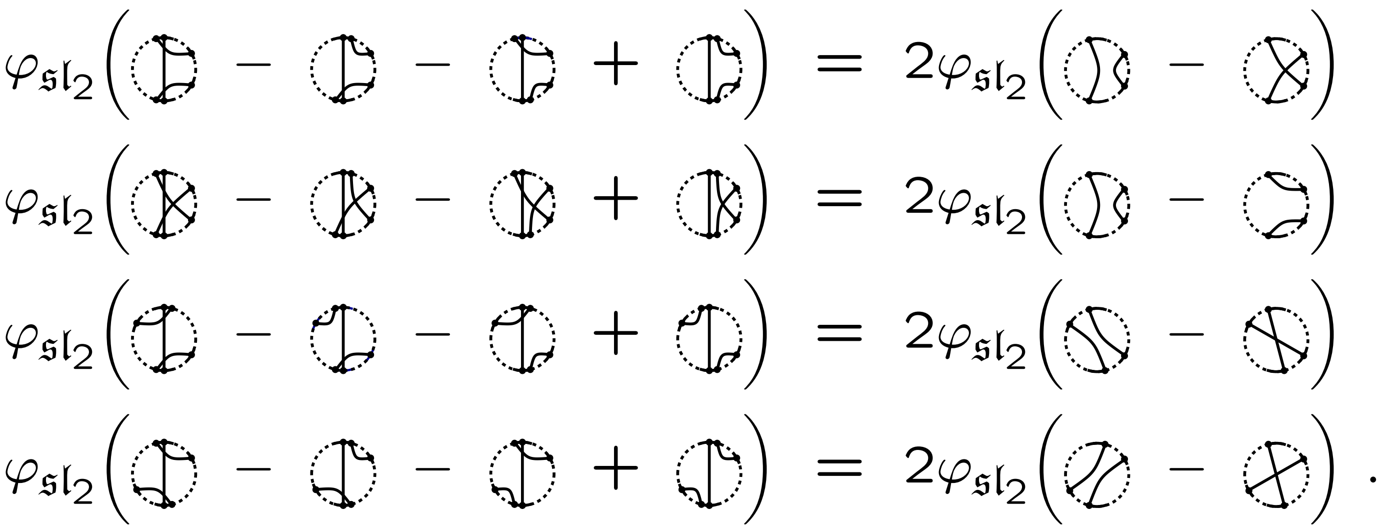
$$\varphi_{\mathfrak{sl}_2,\lambda}(D) = \Big(c_\lambda - \frac{2k}{\lambda}\Big)\varphi_{\mathfrak{sl}_2,\lambda}(D_a) + \frac{2}{\lambda}\sum_{1\le i<j\le k}\Big(\varphi_{\mathfrak{sl}_2,\lambda}(D_{i,j}^{\parallel}) - \varphi_{\mathfrak{sl}_2,\lambda}(D_{i,j}^{\times})\Big).$$

If $k = 1$, the second summand vanishes. In particular, for the Killing form ($\lambda = 4$) and $k = 1$,

$$\varphi_{\mathfrak{g}}(D) = (c - 1/2)\varphi_{\mathfrak{g}}(D_a).$$

It is interesting that the last formula is valid for any simple Lie algebra $\mathfrak{g}$ with the Killing form and any chord a which intersects precisely one other chord.

Lemma (6-term relations for universal $\mathfrak{sl}_2$ wt system)
Let $\varphi_{\mathfrak{sl}_2}$ = weight system for $\mathfrak{sl}_2$ with invariant form $\langle \cdot, \cdot \rangle$;



Proof. ♡.

Remark. These relations also give recursive computation of $\varphi_{\mathfrak{sl}_2}(D)$, as right-hand side two chord diagrams have one chord less than left-hand side; on left-hand side, last three diagrams are simpler than first, since they contain less intersections.

1.4 Bar-Natan representation vs. Kontsevich's

Extending $T : \mathfrak{g} \to \mathrm{End}(V)$ to $U(T) : U(\mathfrak{g}) \to \mathrm{End}(V)$ gives composition

$$\mathcal{A} \overset{\varphi_{\mathfrak{g}}}{\to} U(\mathfrak{g}) \overset{U(T)}{\to} \mathrm{End}(V) \overset{\mathrm{Tr}}{\to} \mathbb{C}$$

as weight system associated with the representation

$$\varphi_{\mathfrak{g}}^{T} = \mathrm{Tr} \circ U(T) \circ \varphi_{\mathfrak{g}}$$

N.B.: • The map $\varphi_{\mathfrak{g}}^{T}$ is not generally multiplicative

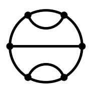
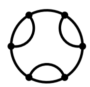
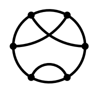
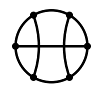
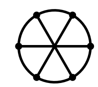
• But if T is irreducible $\Big\{$by to Schur Lemma i.e., $gV \subseteq V, \forall g \in G := GL_N(\mathbb{C})$ irreducible, i.e., $V \subseteq G$ is either 0 or G, then either $V = 0$ or $V = \mathbb{C}^n\Big\}$ then every element of centre $ZU(\mathfrak{g})$ is represented (via $U(T)$) by a scalar operator $\mu \cdot \mathrm{id}_V$, so that its trace equals $\varphi_{\mathfrak{g}}^{T}(D) = \mu \dim V$, and the number $\mu = \frac{\varphi_{\mathfrak{g}}^{T}(D)}{\dim V}$, which is a function of the chord diagram D, is a weight system that is clearly multiplicative.

1.5 Algebra $\mathfrak{sl}_2$ with standard representation

Consider the standard 2-dimensional representation St of $\mathfrak{sl}_2$. Then Casimir element is

$$c = \frac{1}{2}HH + EF + FE = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix} = \frac{3}{2} \cdot \mathrm{id}_2.$$

In degree 3 we have the weight systems:

D					
$\varphi_{\mathfrak{sl}_2}(D)$	c^3	c^3	$c^2(c-2)$	$c(c-2)^2$	$c(c-2)(c-4)$
$\varphi_{\mathfrak{sl}_2}^{St}(D)$	27/4	27/4	−9/4	3/4	15/4
$\varphi_{\mathfrak{sl}_2}^{\prime St}(D)$	0	0	0	12	24

where the last row is the unframed weight system.
And, it comes from the fact that the space $\mathcal{W}$ of unframed weight systems is dual to the space $\mathcal{A}$ of unframed chord diagrams (just as the space $\mathcal{W}^{fr}$ of framed weight systems is dual to the space $\mathcal{A}^{fr}$ of framed chord diagrams)

And, $\mathcal{A}$ is quotient of $\mathcal{A}^{fr}$ by being the subspace spanned by all diagrams with an isolated chord.

So that in terms of multiplication in $\mathcal{A}^{fr}$, $\mathcal{A}$ is described as the ideal of $\mathcal{A}^{fr}$ generated by Θ, the chord diagram with one chord, i.e.,

$$\mathcal{A} = \mathcal{A}^{fr}/(\Theta).$$

And so, we obtain $\varphi'^{St}_{\mathfrak{sl}_2}$ by deframing $\varphi^{St}_{\mathfrak{sl}_2}$ using some n procedure / explicit formula:

$$p_n(D) := \sum_{J \in D} (-\Theta)^{n-|J|} \cdot D_J \ ,$$

• For all $J \subseteq [D]$
• $[D]$ as the set of chords in diagram D
• D_J as subdiagram of D with only chords left from J,
• And, $p : \mathcal{A}^{fr} \to \mathcal{A}^{fr}$ with a sum taken over all n, with $p_n : \mathcal{A}^{fr}_n \to \mathcal{A}^{fr}_n$.

1.6 Algebra $\mathfrak{gl}_N$ with standard representation

Consider standard rep of Lie algebra $\mathfrak{g} = \mathfrak{gl}_N$. Fixing trace of matrices-product as ad-invariant form: $\langle x, y \rangle = \mathrm{Tr}(xy)$, with algebra of $\mathfrak{gl}_N$ being linearly spanned by matrices $e_{ij} = 1,\ \forall\, i = j$, and 0 everywhere else.

$$\langle e_{ij}, e_{kl} \rangle = \delta_i^l \delta_j^k \qquad (\delta = \text{Kronecker delta}).$$

Therefore, duality between $\mathfrak{gl}_N$ and $(\mathfrak{gl}_N)^*$ defined by $\langle \cdot, \cdot \rangle$ is given by $e_{ij}^* = e_{ji}$.

One can verify $[e_{ij}, e_{kl}] \neq 0$ only in the following cases:

- $[e_{ij}, e_{jk}] = e_{ik}$, if $i \neq k$
- $[e_{ij}, e_{ki}] = -e_{kj}$, if $j \neq k$
- $[e_{ij}, e_{ji}] = e_{ii} - e_{jj}$, if $i \neq j$

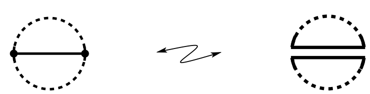
Hence, Lie bracket is given by tensor in $\mathfrak{gl}_N^* \otimes \mathfrak{gl}_N^* \otimes \mathfrak{gl}_N$:

$$[\cdot\,,\cdot] = \sum_{i,j,k=1}^{N} (e_{ij}^* \otimes e_{jk}^* \otimes e_{ik} - e_{ij}^* \otimes e_{ki}^* \otimes e_{kj}).$$

When transferred to $\mathfrak{gl}_N \otimes \mathfrak{gl}_N \otimes \mathfrak{gl}_N$ by above duality, the tensor becomes

$$J = \sum_{i,j,k=1}^{N} (e_{ji} \otimes e_{kj} \otimes e_{ik} - e_{ji} \otimes e_{ik} \otimes e_{kj}).$$

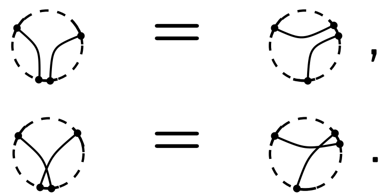
Theorem III (D. Bar-Natan's computation of w.s. $\varphi^{St}_{\mathfrak{gl}_N}$)
Let $s(D)$ be the number of connected components of curve obtained by doubling all chords of a chord diagram D; i.e.,

 .

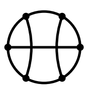
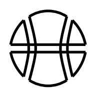
Then $\varphi^{St}_{\mathfrak{gl}_N}(D) = N^{s(D)}$.

Proof (hint). By matrices (e_{ij}) as chosen basis of $\mathfrak{gl}_N$, consider curve γ obtained by doubling the chords with ends labeled e_{ij} and e_{ji} in Wilson loop.

Remark. By definition, $s(D) = c - 1$, where $c =$ number of boundary components of surface satisfying the 2-term relations:



Derivation IX

For $D =$ , we have , $s(D) = 2$, $\varphi^{St}_{\mathfrak{gl}_N}(D) = N^2$.

Proposition. *The weight system $\varphi^{St}_{\mathfrak{gl}_N}(D)$ depends only on the intersection graph of D.*

Proof. $\varphi^{St}_{\mathfrak{gl}_N}(D)$ is defined by $s(D) = c - 1$, therefore a function of the genus of diagram D, where the genus depends only on the intersection graph (although aought to be proved).