On the Quantum-theoretic Empirical Path Integrals and Option Pricing

Matt' Bernard [1†]
mattb@berkeley.edu
Supervisor: Steve Evans [2†]
evans@stat.berkeley.edu

1 Spring 2006, Stat 251: Stochastic analysis with applications to mathematical finance
2 Professor, Statistics & Mathematics (joint appointment)
† University of California, Berkeley, CA 94720, USA

Abstract

This paper surveys family of stochastic quantum-theoretic models over financial variables, alongside option pricing model in the quantum scaling by Quantum Path Integral (QPI). Considering time and space intervals on random volatile paths, we show the QPI model has nice features, notably exotic and path-dependent, allowing for analytic pricing solutions of many options (derivatives) which are canonically, continuous-time jump Markov process. We investigate the classical (stochastic) scheme (with Markov law) generalizing the framework of Quantum Langevin representations and presenting useful expressions that characterize options as probabilistic expectations over $\mathbb{C}$-nilpotent functors.

Keywords: Empirical-QFT, Stochastic-path-integral, Option-pricing

Introduction

Since Mathematical Physics inception of incidental evolution by Brownian motion (Langevin 1908), Langevin equation has become a cosmopolitan tool for description of motion in the presence of noise (N.G. van Kampen 1981). In classical mechanics, Langevin equation received an exact formulation within the framework of stochastic processes, where the classical (analytic) trajectory is replaced by a statistical measure on a set of volatile (non-analytic) trajectories (Wiener 1923). The Langevin equation is thus better seen as stochastic differential equation, for which stochastic integro-differential calculus was developed (Itô 1951, Schuss 1980) to generalize the classical resolution of equations of motion. Therefore, quantum-theoretic method on Langevin stochastic framework is seen as necessary to revise description and formulate corresponding quantum Langevin equation (Mori 1965, Ford et al 1965), leading to quantum syllogism of stochastic processes, non-commuting stochastic processes (Lewis and Thomas 1975, Davies 1976), and of associated calculus (Hasegawa and Streater 1983, Hudson and Parthasarathy 1984). In particular, the last augmentation accounts for both: volatility having its origin in purely statistical aspects of the problem (the presence of noise) and volatility resulting from purely quantum aspects, where formalism mixes measures and operators over $\mathbb{C}$.

Over the twentieth century, quantum-theoretic method continously underwent revision, progressively establishing closer connection to stochastic processes. An efficient representation was given by Feynman (1948) in terms of path integrals. Although defined in less rigorous form, path integrals were already used to generalize classical trajectories to randomly volatile paths. Used to describe a small system coupled to a bath (Feynman and Vernon 1963), path integrals particularly, appear well suited for treating simultaneous pure quantum effects and statistical effects such as like tunneling and dissipation (Caldeira and Leggett 1981). However, in view of connection between operatorial (non-commutative) quantum-theoretic representations and classical (commutative) statistical mechanics, the question naturally arises: Can quantum Langevin equation be re-formulated within a purely classical stochastic framework?

Indeed, similar to Brownian motion, in quantum motion we have noise; for instance, noise induced by the vacuum in the background field hypothesis (Nelson 1985). As a result, a formulation of quantum Langevin equation in purely classical stochastic framework would be the result of a formalism where fluctuations due to noise induced by the "bath" and fluctuations due to the "bath" quantum effect are treated the same way; that is, where only measures appear, instead of measures and operators. Devoted to this approach were DeWitt (1957) and McLaughlin & Schulman (1971) who generated all desired quantum correlation functions (that is, the rigorous form of related Feynman path integrals) from a purely quantum treatment of the physical events' (stochastic) Langevin equation. This includes the quantum representation of stochastic process for both the bath (taken to be infinite set of harmonic oscillators) and the small system, so that: general coupling between positions and velocities of both particle and field source could be measured, and additional Itô terms could appear. Thus, quantum formalism of stochastic processes allows the adjoining of extra Itô terms and provides stochastic integrals necessary for expressing the contribution of non-analytic (non-differentiable) trajectories (Caldeira and Leggett 1981, Ford et a1 1985, Ford and Kac 1986, Nakazawa 1986, and others). In addition, general condition on coupling constants could be obtained for recovering processes which evolve locally in time in the limit of bath consisting of continuum of fields, where resulting stochastic processes are affected by two renormalizations.

By functional models (Glimm and Jaffe 1981) of path integrals on stochastic processes, and further development (Nelson 1966) of path integrals, the connection between classical and quantum-theoretic representations of stochastic processes is further improved and tends to show that the latter is more than an alternative formal description, but constitutes a genuine approach to nonlinearity in stochastic phenomena, including complex financial models. Thus, Baaquie et al. (2002) exploit the approach: Quantum, that is, path integral, formalism of simple models and nonlinear models, including exotic and path-dependent models based on the (stochastic) Langevin equation.

Langevin Evolution with No Stochastic Volatility

Options pricing, as in [1], is typically a non-linear function of underlying asset and some other variables, such as interest rates, strike, et ceteras. And, the basis of any options pricing model is an evolution of random walk phenomenon characterized by the (stochastic) Itô-Weiner process (or Langevin equation) followed by the underlying asset on which the option is written. Recalling the definition of Wiener process (or Brownian motion), we have as follows:

Definition 0.0.1. *An $\mathbb{R}$-valued stochastic process $W(\cdot)$ is Brownian motion or Wiener process if:*

1. $W(0)$ *is* 0 *a.s.*
2. $W(t) - W(s)$ *is* $(0, t-s)$ $\forall t \geq s \geq 0$
3. *for all times* $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$*, the random variables are independent ("independent increments"). In particular* $\mathbb{E}[W(t)] = 0$, $\mathbb{E}[W^2(t)] = t$ *for each time* $t > 0$.

From basic principles of finance (given in [1]), underpinning theory of security derivatives, in particular, theory of options pricing, it is clear the call option value $f(T)$, when the option matures at time $t = T$, is given by:

$$f(T, S(T)) = \begin{cases} S(T) - K, S(T) > K \\ \\ 0, S(T) < K \end{cases} = g(S) \tag{1}$$

In addition, the evolution of the security $S(t)$ is modeled as random (stochastic) variable given by a stochatic Langevin equation, the Itô-Wiener process:

$$\frac{dS(t)}{S(t)} = \phi dt + \sigma R(t) dt \tag{2}$$

where R is the usual Gaussian white noise with zero mean, and uncorrelated values at time t; ϕ is the drift term or expected return, while σ is the volatility.

Here, white noise is assumed to be independent for each time t. Thus, the Dirac delta-function correlator given by:

$$t' \langle R(t) R(t') \rangle = \delta(t - t') \tag{3}$$

Recall, white noise $R(t)$ has the following important property:

Property 0.0.1. *If we discretize time $t = n\epsilon$, then the probability distribution function of white noise is given by: $P(R_t) = \sqrt{\frac{\epsilon}{2\pi}} e^{-\frac{\epsilon}{2} R_t^2}$*

Thus, for random variable R, it can be shown that $R_t^2 = \frac{1}{\epsilon} +$ random terms of $O(1)$.

In other words, to leading order of ϵ, the random variable R_t^2 becomes deterministic. This property of white noise leads to a number of important results, and goes under the name of Itô calculus in probability theory.

In the Black-Scholes model, the assumption is that the stock price is log-normally distributed [that is, the price follows a Wiener process with constant drift (the average growth rate of the underlying security) and constant variance (the volatility)]. Explicitly, this is shown by applying the Itô lemma to (2) yielding,

$$\partial f = \frac{\partial f}{\partial S} dS + \left(\frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 f}{\partial S^2} \right) dt = \left(\frac{\partial f}{\partial S} + \frac{\partial f}{\partial S} \frac{\sigma^2 S^2}{2} \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S R \frac{\partial f}{\partial S} dt \tag{4}$$

and eliminating the stochastic term by considering the portfolio $\pi = f - \frac{\partial f}{\partial S} S$ (since we cannot value (4) directly), we see that:

$$d\pi = df - \frac{\partial f}{\partial S} dS = \left(\frac{\partial f}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 f}{\partial S^2} \right) dt = r\pi dt = r \left(f - S \frac{\partial f}{\partial S} \right) dt \tag{5}$$

with the last equality stemming from the no-arbitrage condition. Thus, the no stochastic term π is a risk-free investment and hence must offer the same return as any other risk-free investment.

Remark 0.0.1. *In fact, it is in simplifying the above equation that the black-scholes equation itself is obtained as:*

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{6}$$

Quantum-theoretic Formalism

The Black-Scholes equation (where there is no stochastic volatility) in (6) is recast into a Schrödinger-like equation f. The value of the option, is identified as a wave function dependent on time and the price of the underlying security. Thus, the Schrödinger equation becomes:

$$\frac{\partial f}{\partial t} = \left(\hat{H}_{BS} + r \right) f \tag{7}$$

where the Hamiltonian, $\hat{H}_{BS}$, is given by:

$$\hat{H}_{BS} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rS\frac{\partial f}{\partial S} \tag{8}$$

and introducing the variable $x = \ln S$ for subsequent simplification, the Hamiltonian, $\hat{H}_{BS}$, becomes

$$\hat{H}_{BS} = -\frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \left(\frac{\sigma^2}{2} - r\right)\frac{\partial}{\partial x} \tag{9}$$

Solution by Quantum-theoretic Method

First, we recall the Risk-neutral Valuation Method for comparison later.

Recall 0.0.1. *An elegant solution using minimal mathematics, Risk-neutral Valuation applies Itô lemma to* (2)*, giving:*

$$d(\ln S) = \left(\phi - \frac{\sigma^2}{2}\right) + \sigma R \tag{10}$$

which implies the present value c of the option is the expected final value $\mathbb{E}[\max(S-K,0)]$ of the option discounted at the risk-free interest rate:

$$c = e^{-r(T-t)}\mathbb{E}[\max(S-K,0)] = e^{-r(T-t)}\int_K^\infty (S-K)g(S)d(S) \tag{11}$$

where $g(S)$, the probability density, is given by

$$g(S) = \frac{1}{\sigma S\sqrt{2\pi(T-t)}} exp\left(-\frac{\left(\ln\left(\frac{S}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right) \tag{12}$$

(ϕ replaced by r in accordance with the principle).

Intuitively,

$$c = e^{-r(T-t)}[e^{r(T-t)}SN(d_1) - KN(d_2)] \tag{13}$$

where

$$d_1 = \frac{\ln(\frac{S}{K}) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{14}$$

$$d_2 = \frac{\ln(\frac{S}{K}) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T} \tag{15}$$

such that $N(d_2)$ is probability that the final stock price will be above K (i.e., option will be exercised) in a risk-neutral world and $KN(d_2)$ is strike price multiplied by probability that strike price will be paid.

In addition, expected value of a variable that equals S is the expression: $SN(d_1)e^{r(T-t)}$ if $S \geq K$, & 0 otherwise, in risk-neutral world. And, $e^{r(T-t)}SN(d_1) - KN(d_2)$ is expected value of the option at maturity).

Moreover, integrating the white noise R with respect to the times gives the random walk distribution which is known to be normal. In fact, from (10), it can be seen that

$$\ln S - \ln S_0 \sim N\left[\left(\phi - \frac{\sigma^2}{2}\right)(T-t),\, \sigma\sqrt{T-t}\right] \tag{16}$$

or

$$\ln S \sim N\left[\ln S_0 + \left(\phi - \frac{\sigma^2}{2}\right)(T-t),\, \sigma\sqrt{T-t}\right] \tag{17}$$

(S and S_0 being prices of the underlying security at time T and t respectively)

Now, we introduce, yet, another elegant way of solving the Black-Scholes equation, still using minimal mathematics, but with quantum-theoretic formulation. From [1], quantum-theoretic formulation of the case of constant volatility (which does not have any much advantage over the risk-neutral valuation method) has the value of the option (that is, the price) to be given by the Feynman-Kac formula:

$$f(t,x) = \int_{-\infty}^{\infty} dx' \langle x|e^{-(T-t)H_{BS}}|x'\rangle g(x',K) \tag{18}$$

where K denotes the strike price, $g(x',K)$ the payoff function given by

$$g(x',K) = max(e^{x'} - K, 0) \tag{19}$$

Changing to "momentum" basis where the Hamiltonian $\hat{H}_{BS}$ is diagonal, the transformation from "position" basis is given as follows:

$$\langle x|x'\rangle = \delta(x-x') = \int_{-\infty}^{\infty} \frac{dp}{2}\pi d^{ip(x-x')} = \int_{-\infty}^{\infty} \frac{dp}{2}\pi \langle x|x'\rangle\langle p|x'\rangle \tag{20}$$

such that the $\hat{H}_{BS}$ of the momentum basis is defined by

$$\langle p|\hat{H}_{BS}|p'\rangle = \left[\frac{\sigma^2 p^2}{2} + ip\left(\frac{\sigma^2}{2} - r\right)\right]\delta(p-p') \tag{21}$$

Thus,

$$\langle x|e^{-\tau\hat{H}_{BS}}|x'\rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\langle x|p\rangle\langle p|e^{-\tau\hat{H}_{BS}}|p'\rangle\langle p'|x'\rangle \tag{22}$$

$$= \int_{-\infty}^{\infty} \frac{dp}{2\pi} exp\left(-\frac{\tau\sigma^2 p^2}{2} + ip\left[x - x' + \tau\left(r - \frac{\sigma^2}{2}\right)\right]^2\right) \tag{23}$$

$$= \frac{1}{\sigma\sqrt{2\pi\tau}} exp\left(-\frac{1}{2\tau\sigma^2}\left[x - x' + \tau\left(r - \frac{\sigma^2}{2}\right)\right]^2\right) \tag{24}$$

where $\tau = T - t$

Again, changing variables from x to S (noting that time runs backward here), it is evident that the distribution (solution) obtained here is the same as (12) as it should be. Hence, for a non-stochastic volatility, the quantum-theoretic method is essentially the same as risk-neutral valuation method as it should be.

Langevin Evolution with Stochastic Volatility

Recall 0.0.2. *As in the current paper, recall the classical Merton-Garman model (a generalized Black-Scholes model). That is, processes (in the presence of stochastic volatility) driven by two correlated white noise functions R_1 and R_2 given by:*

$$\begin{cases} \frac{dV}{dt} = \lambda + \mu V + \zeta V^{\alpha} R_1 \\ \frac{dV}{dt} = \phi S + \sigma\sqrt{V} + \mu V + \zeta V^{\alpha} R_2 \end{cases} \tag{25}$$

such that $V = \sqrt{\sigma}$, and $\langle R(t)R(t')\rangle = \frac{1}{p}\delta(t-t')$ being the correlation parameter for $-1 \leq \rho \leq 1$

The evolution of the option price in the presence of stochastic volatility does amount to: The implicit choice of $\beta(S, V, t, r)$, the market price of volatility risk (since the higher the value of β the more averse the investor are to take on the volatility risk), and the modeling of β appropriately from the portfolio

$$\Pi = f_1 + \Gamma_1 f_2 + \Gamma_2 S \tag{26}$$

such that

$$d\Pi = (\Theta + \Gamma_1\Theta_2 + \Gamma_2\phi S)dt + (\Xi_1 + \Gamma_1\Xi_2 + \Gamma_2\sigma S)dX + (\Psi_1 + \Gamma_1\Psi_2)dY \tag{27}$$

Setting

$$\begin{cases} \Xi_1 + \Gamma_1\Xi_2 + \Gamma_2\sigma S = 0 \\ \Psi_1 + \Gamma_1\Psi_2 = 0 \end{cases} \tag{28}$$

We obtain

$$\begin{aligned} \Gamma_1 &= -\frac{\Psi_1}{\Psi_2} = -\frac{\partial f_1/\partial V}{\partial f_2/\partial V} \\ \Gamma_2 &= \frac{\partial_1}{\partial_2}\frac{\partial f_2}{\partial S} = \frac{\partial f_1/\partial V}{\partial f_2/\partial V}\frac{\partial f_2}{\partial S} - \frac{\partial f_1}{\partial S} \end{aligned} \tag{29}$$

Then, expanding

$$d\Pi = r\Pi dt \tag{30}$$

We obtain β as:

$$\begin{aligned} &\beta(S, V, t, r) \\ &= \frac{1}{\partial f_1/\partial V}\left(\frac{\partial f_1}{\partial t} + (\lambda + \mu V)\frac{\partial f_1}{\partial V} + rS\frac{\partial f_1}{\partial S} + \frac{VS^2}{2}\frac{\partial^2 f_1}{\partial S^2} + \rho V^{1/2+\alpha}\xi\frac{\partial^2 f_1}{\partial S\partial V} + \frac{\xi^2 V^{2\alpha}}{2}\frac{\partial^2 f_1}{\partial S\partial V} - rf_1\right) \\ &= \frac{1}{\partial f_2/\partial V}\left(\frac{\partial f_2}{\partial t} + (\lambda + \mu V)\frac{\partial f_2}{\partial V} + rS\frac{\partial f_2}{\partial S} + \frac{VS^2}{2}\frac{\partial^2 f_2}{\partial S^2} + \rho V^{1/2+\alpha}\xi\frac{\partial^2 f_2}{\partial S\partial V} + \frac{\xi^2 V^{2\alpha}}{2}\frac{\partial^2 f_2}{\partial S\partial V} - rf_2\right) \end{aligned} \tag{31}$$

Remark 0.0.2. *The question* **why is the β parameter needed for the stochastic volatility options princing and not for Black-Scholes options pricing** *might be posed.*

The answer is not far-fetched. Volatility is not traded in the market. Hence, it is not possible to perfectly hedge against the underlying security price. Thus, investor-risk preferences have to be taken into account when considering stochastic volatility. In other words, risk-neutral valuation cannot be applied directly to volatility since volatility is directly traded in the market.

Now, the estimation of β, however, could be unyielding; yet, Lamoureux, et al.[24] gave some evidence that β is nonzero. Consequently, assuming Cox-Ingresoll-Ross model (where spot-asset return is constantly correlated with "consumption," giving rise to a risk premium which is proportional to volatility) for simplicity as it has the effect of redefining μ in above-given equation, the market price of risk is thus included in the Merton & Garman equation by redefining μ. Hence the equation of Merton & Garman for the contemporary process is

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + (\lambda + \mu V)\frac{\partial f}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 f}{\partial S^2} + \rho\xi V^{\frac{1}{2}+\alpha}\frac{\partial^2 f}{\partial S \partial V} + \xi^2 V^{2\alpha}\frac{\partial^2 f}{\partial V^2} = rf \tag{32}$$

Letting $x := \ln S$ and $y := \ln V$, the Merton-Garman equation becomes:

$$\frac{\partial f}{\partial t} + \left(r - \frac{e^y}{2}\right)\frac{\partial f}{\partial x} + \left(\lambda e^y + \mu\frac{\xi^2}{2}e^{2y(\alpha-1)}\right)\frac{\partial f}{\partial y} + \frac{e^y}{2}\frac{\partial^2 f}{\partial x^2} + \rho\xi e^{y(\alpha-1/2)}\frac{\partial^2 f}{\partial x \partial y} + \xi^2 e^{2y(\alpha-1)}\frac{\partial^2 f}{\partial y^2} = rf \tag{33}$$

where $f(T) = \max(e^x - K, 0)$. Thus, bringing us to the final value problem.

Classical Solution

The "straightforward" solution can be obtain for $\rho = 0$, regardless of if volatility is stochastic or not. First, recalling the following theorem:

Theorem 0.0.1. "Merton Theorem". *The solution for a deterministic volatility process is the Black-Scholes price with the volatility variable replaced by the average volatility.*

And, considering that the stochastic case is a collection of a large number of deterministic volatility processes, option price is hence the average of prices for each of the processes. That is, for a volatility that follows the generic process $V(t)$, where V may be stochastic, the option price is given by:

$$C = \int_0^\infty [SN(d_1(V)) - Ke^{-r\rho}N(d_2(V))]V_m(V)dV \tag{34}$$

where V_m is the probability distribution function for the volatility mean (which in fact is a delta function for a deterministic process) such that $d_1(V)$ and $d_2(V)$ are the same as the variables defined in (14) and (15). In particular,

Case 0.0.1. *Considering a deterministic process*

$$V = V_0 e^{\mu t}, \quad 0 \le t \le T \tag{35}$$

the probability density function for volatility mean is given by

$$V_m = \delta\left(V - V_0\frac{e^{\mu t} - 1}{\mu T}\right) \tag{36}$$

Case 0.0.2. *Considering a stochastic volatility process chosen to be*

$$dV = \xi Q dt, V(0) = V_0, 0 \le t \le T \tag{37}$$

where Q represents white noise, the distribution of V-mean during the time interval $(0, T)$ is given by

$$V_m \sim N\left(V_0, \frac{\xi^2 T}{3}\right) \tag{38}$$

Thus, option price is given by

$$C = \sqrt{\frac{3}{2\pi\xi^2 T}}\int_0^\infty [SN(d_1(V)) - Ke^{-r\tau}N(d_2(V))]exp\left(\frac{3(V - V_0)^2}{2\mu^2 T}\right)dV \tag{39}$$

The latter can be computed numerically in a fraction of a second, while distribution of the volatility mean is obtained by Monte-Carlo simulation of the stochastic process; thus, finding the average volatility and the Black-Scholes price using the average volatility at each step.

A Quantum-theoretic Solution

The Merton-Garman-Schrödinger equation (a Hamiltonian formulation of an evolution of option price in the presence of stochastic volatility) given by

$$\frac{\partial f}{\partial t} = (r + \hat{H}_{MG}(x,y))f(x,y,T) = \max(e^x - K, 0) \tag{40}$$

where

$$dV = (\lambda + \mu V + \xi V^\alpha Q)dt \tag{41}$$

with Hamiltonian

$$\hat{H}_{MG} = -\left(r - \frac{e^y}{2}\right)\frac{\partial}{\partial x} - \left(\lambda e^{-y} + \mu - \frac{\zeta^2}{2}e^{2y(\alpha-1)}\right)\frac{\partial}{\partial y} - \frac{e^y}{2}\frac{\partial^2}{\partial x^2} - \rho\zeta e^{y(\alpha-1/2)} \tag{42}$$

has the general solution given by

$$\begin{cases} f(x,y,t) = e^{-r\tau}\int_{-\infty}^{\infty} dx' \langle x,y|e^{-\hat{H}\tau}|x'\rangle f(x',T) \\ \\ f(x,t) = \max(e^x - K, 0), \tau = T - t \end{cases} \tag{43}$$

The special case $\alpha = 1/2$ was solved by Hull and White[26] using the series method, and by Heston[27] using elementary probability techniques. The special case $\alpha = 1$ was solved by Baaquie, et al.[28] using the path integral technique of quantum theory.

Lagrangian Formulation

Central to the question of quantum-theoretic solution is the Lagrangian formulation approach as follows. As mention in [1], time is discretized to maturity so that there are N time steps, with interval $\epsilon = \frac{\tau}{N}$. By decomposition, the propagator operator is given by the pricing kernel

$$P(x,y,T|x',t) = \int DX_{BS}e^{S_{BS}} = \langle x,y|e^{-\hat{H}\tau|x',y'}\rangle \tag{44}$$

or

$$P(x,T|x',t) = \int DX_{BS}e^{S_{BS}} = \langle x,y|e^{-\hat{H}\tau|x',y'}\rangle \tag{45}$$

hence, the option price, just as in the Black-Scholes case where the only variable is x (y being constant). Thus, in the Black Scholes model for instance, Baaquie, et al. arrived at the action

$$S_{BS} = \epsilon\sum_{i=1}^{N} L_{BS}(i) \tag{46}$$

with

$$L_{BS}(i) = -\frac{1}{2\sigma^2}\left(\frac{x_i - x_{i-1}}{\epsilon} + r - \frac{\sigma^2}{2}\right)^2 \tag{47}$$

where discretized positions (x_i) are introduced for the variable $x = log(S)$ which distinguishes the quantum state of the system.

Since the action is quadratic, the integral of the action over the stock price is taken as follows

$$Q = \int DX_{BS} = \prod_{t=0}^{r}\int_{-\infty}^{\infty} dx(t) \tag{48}$$

Similarly, with the exception that a generic potential is introduced, the barrier options is obtained as

$$H_v = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - V(x)\right)\frac{\partial}{\partial x} + V(x) \tag{49}$$

Monte Carlo Simulation

The pricing kernel, which is the fundamental quantity to compute here is found using functional integral, Montagna et al.[27]. Baaquie, et al., however, used the standard metropolis algorithm: *"Where thermalization is slow," Baaquie, et al noted, "sequentially Metropolis updates and cluster updates can be used, with the latter being an update for the embedded Ising dynamics in the lattice variables $x_i/|x_i|$, and included in the faster generation of the thermalized paths of the stock price $x(t)$."*

Computing the option price directly, rather than by the propagator itself, is less cumbersome. Thus, denoting the payoff function (in the option pricing equation given by the Feynman-Kac formula (18)) with $g(x, K)$, where K is strike price, Baaquie, et al. carried out a simulation by fixing an initial point x and the letting final point evolve according to the quantum dynamics, such that a path (x, x') is generated; and, x is allowed to undergo some quantum fluctuations at the fixed x, such that each x is convoluted with the payoff function and then averaged. And, the procedure is repeated such that the option price is obtained at time to maturity τ.

For barrier options specifically, Baaquie, et al. used the Langevin method in the presence of a step potential sitting at a value of the stock price given by $x_0 = log(S_0)$, with $S_0 = 100$, which in effect implies that a price has been discounted compared to the Black-Scholes.

Conclusion

To the astute reader, we used quantum theory from Mathematical Physics in finance to affirm the hybrid *Quantum Finance*. The Quantum Path Integral (QPI) technique in particular, has produced fruitful results in solving the problem of pricing options when the volatility is stochastic with a correlation for stock price. It has given rise to analytic pricing solutions such as a the full solution for the volatility process of (42). By the approach of Baaquie et al. we found useful expression (44) which enables us to solve the problem of option price of a general volatility process as an expectation over certain functionals on the probability space spanned by the myriad volatility paths. The next step, which is to derive the distribution functions for these functionals, is hence to be accomplished; that is, an analytic solution for a general volatility process. Considering that path integral techniques in finance are very new, these successes are not insignificant.

Analytic solutions, are, of course, only one side of the coin. If there are efficient numerical algorithms to find the solution, the lack of an analytic solution does not constitute a serious hindrance. As shown, extreme efficient Monte Carlo methods can be found by the QPI methods since a large number of degrees of freedom has been integrated out. Thus, the propagator for the stock prices, which is not so easily done, can also be directly investigate using more standard Monte Carlo techniques.

The empirical question of whether these stochastic volatility processes can adequately describe the market is somewhat more difficult to state (rather than to solve). Part of this difficulty is defining what the initial volatility actually is. The shortest time period over which average volatility figures are available is 10 days. This means the initial volatility is known only in very imprecise manner; and, the situation gets worse when considering that most option trading occurs just few days before maturity. However, the time period over which volatility data is collected and distributed by exchanges and financial information agencies (such as Reuters and Bloomberg) has been set by convention and is really not binding. Thus, it is still possible to directly analyze the trades to obtain short term volatility information. But, any meaningful empirical study will necessarily be large-scale and involve considerable effort; preliminary tests such as the one done by Baaquie, et al. elsewhere, and the fact that relatively general volatility processes can be studied using path integral techniques suggest that it is not unlikely that positive results might be achieved in this resolution.

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