On the Quantum-theoretic Empirical Investigation of Forward Rates

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Abstract

With respect to the formal (universal) description of volatile classical (stochastic) paths for quantum field theoretic (QFT) models, we investigate stochastic quantum field theory of classical (stochastic) processes $(\mathcal{F}_T, \mathcal{P})$. In particular, for arbitrary $\mathcal{F}_T$-measurable spaces and processes, we describe the emerging stochastic quantum family and probability measure spaces of forward rates by affine term structure in quantum field theoretic (QFT) framework.

Keywords: Empirical-QFT, Affine-term-structure, Forward-rates

1 Introduction

Inspired by metaphysical developments in quantum field-theoretic modeling of volatility, quantum field-theoretic models have been applied to classical models of forward rates (that is, interest rates), in particular, Heath-Jarrow-Morton (HJM) model. However, several empirical tests with the classical HJM model in the quests by researchers such as: Bühler, UhrigHomburg, Walter and Weber [15], Flesker [16], Sim and Thurston [17]), proved abortive: all of the tests assume a certain form for the volatility function σ. Hence, there is need for a test which is independent of the volatility function.

Reviewing some basic stuffs:

Definition 1.0.1. *Interest rates at any point in time form a usually continuous curve (current interest rates for different times in the future) called the forward rate curve (FRC). A stochastic variable with very minimal loss of generality, the forward rate is denoted by $f(t, x)$, which represents the interest rate at future time x for a contingent T-claim entered into at time $t < x$. For example, $f(1, 2)$ is the interest rate one year from now for an instantaneous deposit to be made 2 years into the future.*

Definition 1.0.2. *Bonds, the financial instruments of debt issued by governments and corporations to raise money from the capitals market, have a pre-determined (deterministic) cash flow (i.e., a contingent T-claim).*

Bringing the notion of contingent claim to limelight here:

Definition 1.0.3. *A contingent T-claim is any random variable $X \in L^0(\mathcal{F}_T, \mathcal{P})$ (i.e., an arbitrary $\mathcal{F}_T$-measurable random variable). The notation $X \in L^0_+(\mathcal{F}_T, \mathcal{P})$ denotes set of non-negative elements of $L_0(\mathcal{F}_T, \mathcal{P})$, and $X \in L^0_{++}(\mathcal{F}_T, \mathcal{P})$ denotes set of elements X of $L^0_+(\mathcal{F}, \mathcal{P})$ with $P(X > 0) > 0$.*

In addition:

Definition 1.0.4. *A probability measure Q is a martingale measure if*

1. $Q \sim P$,

2. The discounted price process Z is a Q-local martingale.

If the discounted price process Z is Q-martingale, we say that Q is a strong martingale measure.

And, with the concept of a martingale measure:

Definition 1.0.5. *A self-financing portfolio h such that the corresponding value process has the properties:*

1. $V(0) = 0$

2. $V(T) \in L^0_{++}(\mathcal{F}, P)$

is an arbitrage portfolio.

And if no arbitrage portfolios exists for any $T \in \mathbb{R}_+$, then the model is said to be "free of arbitrage" or "arbitrage free."

Furthermore, by a slight modification of the set of admissible portfolios:

Definition 1.0.6. *A self-financing portfolio h is called Q-admissible if $V^Z(t, h)$ is a Q-martingale for a given martingale measure Q*

Remark 1.0.1. *By definition Z is a Q-martingale, V^Z-process is the stochastic integral of h with respect to Z. Thus, it is clear that every sufficiently self financing portfolio is in fact admissible. Of course, it could be annoying that the definition of admissibility is dependent upon the particular choice of martingale measure, but the need for the admissibility condition can be seen inside the proof (given in* [4]*) of one of the basic results in the theory which states that:*

Property 1.0.1. *A model is free of arbitrage in the sense that there exist no Q-admissible arbitrage portfolio if there exist a martingale measure Q.*

Now, in retrospect, the interpretation of the contingent claim is that a contract which specifies that the stochastic amount, X, of money is to be payed out to the holder of the contract at time T.

As an illustration, zero coupon bonds, also known as pure discount bonds, with maturity date T (i.e. T-bond) are a contract which guarantees a single cash flow consisting of a fixed payoff of say $1 Naira$ at some future (maturity) time T; the price at time t of a bond with maturity date T is denoted by $P(t, T)$.

Thus, given a bond market, a number of interest rates $(S_0, \ldots, S_k)$ can be defined almost surely by a system of stochastic differential equations (SDE), driven by a finite number of Wiener processes, in a defined filtered probability space (Ω, F_t, P) carrying the finite number of stochastic processes (S-processes) which are assumed to be all semi-martingales.

Driven by the Martingale models for the short rate given by:

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t)$$
$$for \begin{cases} \mu = drift\ term \\ \sigma = diffusion\ term \end{cases}$$

The term structure for the systems of stochastic differential equations, SDE (which are some standard models of the sought) are completely determined by specifying the r-dynamics under the martingale measure $\mathcal{Q}$, and are in agreement with the Affine Term Structure (ATS) theory which states that:

Theorem 1.0.1. *A model is said to possess an affine term structure (ATS) if the term structure* $\{P(t,T) \mid 0 \leq t \leq T, \ T \geq 0\}$ *has the form* $P(t,T) = F(t, r(t), T)$*, where* F *has the form*

$$F(t, r, T) = e^{A(t,T) - B(t,T)r}$$

for deterministic functions A *and* B *given by*

$$A(t,T) = \frac{\sigma^2}{2} \int_t^T B^2(t,T)ds - b \int_t^T B(t,T)ds$$
$$B(t,T) = \frac{1}{a}\left(1 - e^{a(T-t)}\right)$$

Thus, we have the following derivations:

1.1 The Vasicek Model

Based on the SDE:

$$dr = (b - ar)dt + \sigma dW$$

the Vasicek model has the property of being mean reverting (under the martingale measure $\mathcal{Q}$) in the sense that it will tend to revert to the mean level b/a. And, with the term structure computed in [5], the price property can be stated as follows:

Property 1.1.1. *The bond prices are given by:*

$$P(t,T) = e^{A(t,T) - B(t,T)r}$$

where

$$B(t,T) = \frac{1}{a}\left(1 - e^{a(T-t)}\right)$$

and

$$A(t,T) = \frac{(B(t,T) - T)\left(ab - \frac{1}{2}\sigma^2\right)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}$$

1.2 The CIR Model

A much more difficult model to handle compared to the Vasicek model, the Cox-Ingersoll-Ross (discussed in depth in [6] and [7]), we have the following property:

Property 1.2.1. *The term structure is given by:*

$$F^T(t, r) = A(T - r)e^{-B(T-r)r}$$

where

$$B(x) = \left(\frac{2(e^{\gamma x})}{(\gamma + a)(e^{\gamma x} - 1) + 2\gamma}\right)^{\frac{2ab}{\sigma^2}}$$

and

$$\gamma = \sqrt{a^2 + 2\gamma^2}$$

1.3 The Hull and White Model

Detailed in [8], the Hull and White model has the following property, consistent with the $\mathcal{Q}$-dynamics of the short rate given by:

$$dr = (\phi(t) - ar)dt + \sigma dW(t)$$

where a and σ are constants, and ϕ is a deterministic function of time, such that a and σ are chosen to fit the nice volatility structure and ϕ is chosen to fit the theoretical bond prices $\{P(0,T)\,|\,T>0\}$ on the evolution curve $\{P^*(0,T)\,|\,T>0\}$.

Property 1.3.1. *The bond prices are given by:*

$$P(t,T) = e^{A(t,T)-B(t,T)r}$$

where A and B solve

$$\begin{cases} B_t(t,T) - aB(t,T) = -1 \\ B(T,T) = 0 \end{cases} \tag{1}$$

$$\begin{cases} \phi(t)B_t(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) = A(t,T) \\ A(T,T) = 0 \end{cases} \tag{2}$$

with the solution given by:

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$$

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B^2(t,T)ds - b\int_t^T B(t,T)ds$$

Thus, it is obvious that

Remark 1.3.1. • *The short rate r is the only (one-factor) explanatory variable in all of these (classical) standard models.*
• *Also, specifying r as the solution of an SDE allows the use Markov process theory, so that work can be done within a PDE framework. In particular,*
it is often possible to obtain analytical formulas for bond prices and derivatives.

However, the drawbacks remain that

Remark 1.3.2. *they only deal with the spot rate (current interest rate for the present time) and the forward rate curve is treated as a derived quantity. And as the short rate model becomes increasingly more realistic, the yield curve inversion described in* [9] *becomes increasingly more difficult. But from an economic point of view, it is quite unreasonable to assume that the entire money market is governed by only one explanatory variable.*
Hence, it is hard to obtain a realistic volatility structure for the forward rates without introducing a very complicated short rate model.

These and other consideration lead to the proposal of the new model - which use more than one state variable - namely: **The Martingale Modeling** [4], **The Musiela Parameterization** [4]. One bright idea however, was to present an a priori model for the short rate as well as for some long rate, so that one or several intermediary interest rates could be modeled. The method proposed by **Health-Jarrow-Morton, HJM** is at the far end of this spectrum.

2 The Classical Heath-Jarrow-Merton Model

Where the entire forward rate curve is the (infinite dimensional) state variable, the assumption here is that

Assumption 2.0.1. *For every fixed $T > 0$, the forward rate $f(\cdot, T)$ has a stochastic differential which under the objective measure $\mathcal{P}$ is given by*

$$\begin{cases} df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t) \\ \\ f(0,T) = f^*(0,T) \end{cases} \tag{3}$$

where W is a (d-dimensional) $\mathcal{P}$-Wiener process where as $\alpha(\cdot, T)$ and $\sigma(\cdot, T)$ are adapted processes.

Thus conceptually from (1), the HJM model is one stochastic differential in the t-variable for each fixed choice of (maturity) T , with the observed forward rate curve $\{f^*(0,T)|T \geq 0\}$ as the initial condition, automatically giving a good fit between observed and theoretical bond prices at $t = 0$, hence tranquilizing the task of inverting the yield curve.

However, observe that

Remark 2.0.1. *The HJM approach to interest rates is not a proposal of a specific model. Rather, it is a framework to be used for analyzing interest rate models.*

Hence, every short rate model can be equivalently formulated in forward rate terms, and for every forward rate model, the arbitrage free price of a contingent T -claim X is still given by the pricing formula

$$\Pi(0, \mathcal{X}) = E\left[exp\left(\int_0^T r(s)ds\right) \cdot \mathcal{X}\right] \tag{4}$$

where the short rate as usual is given by $r(s) = f(s, s)$.

In addition,

Remark 2.0.2. *Specifying α, σ, with $\{f^*(0,T)|T \geq 0\}$, is essentially the same as specifying the entire forward rate structure. In fact, by the relation*

$$P(t,T) = exp\left(-\int_0^T f(t,s)ds\right) \tag{5}$$

the entire term structure $\{P(t,T)|T > 0, 0 \leq t \leq T\}$ is specified.

Thus, based on the HJM drift theory (proved in [4]) we have the following theorem:

Theorem 2.0.1. *There exists a d-dimensional column-vector process*

$$\lambda(t) = [\lambda_1(t), \ldots, \lambda_d(t)]^T \tag{6}$$

with the property

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)^T ds - \sigma(t,T)\lambda(t) \tag{7}$$

$\forall T \geq 0$ & $\forall t \leq T$ *(where $[\cdots]^T$ denotes transpose).*

That is, it is possible to observe how the processes α and σ should be related so that the induced system of bond prices admits no arbitrage possibilities within the d sources of randomness (one for every Wiener process) and infinite number of traded assets (one bond for each maturity T).

Remark 2.0.3. *Hence, it is obvious that* **the Brownian motions on which the HJM model depends are independent of x***; that is, the HJM is limited.*

3 Quantum-theoretic Model of the One-factor Forward Rates

Thus far, the classical models underlying empirical tests done by erstwhile researchers such as Bühler, Uhrig-Homburg, Walter and Weber [15], Flesker [16], Sim and Thurston [17], et ceteras, have been dissected, and the anatomy characterizing their defects have been annotated. Of the classical models, what seems closer to reality is the HJM; provided the reasoning becomes plausible, given that it is representative, if it is logical and if there is empirical support (proof-test) for its representation from the quantum-theoretical model of the classical HJM. Following the Baaquie, et al. model in [1], the forward rate curve as a quantum-theoretical version of the HJM model which originates in [2] is realized as follows:

Assumption 3.0.1. *Assuming the theory is time translation invariant.*

The stated one-factor quantum field-theoretic string of the forward rates is given by:

$$\frac{\partial f(t,x)}{\partial t} = \alpha(t,x)\sigma(t,x)A(t,x) \tag{8}$$

where $A(t,x)$ is a quantum field with the action given by

$$S[A] = \int_{t_0}^{\infty} dt \int_{t}^{t+T_{FR}} dx\mathcal{L}[A] \tag{9}$$

$$\mathcal{L}[A] = -\frac{1}{2}\left(A^2(t,x) + \frac{1}{\mu^2}\left(\frac{\partial A(t,x)}{\partial x}\right)^2\right) \tag{10}$$

and T_{FR} (introduced to ensure the action is well defined and does not affect final results as the limit to infinity is taken) is the largest time-to-maturity for which the forward rates are defined (in the domain of semi-infinite parallelogram given by $t > t_0, t < x < t + T_{FR}$) such that:

- $\sigma(t,x)$ is assumed to be dependent only upon the variable $\theta = x - t$ based on Assumption 3.0.2
- the initial forward rate curve $f(t_0, x)$ is fixed
- the field values of $A(t,x)$ resting on the boundary points of the domain are arbitrary and are integration variables
- the second term in the action given in (7) valid from [12], hence not abrogated by any arbitrage.

Remark 3.0.1. *As* $\mu \to 0$*, Baaquie, et al. in* [5]*, showed that the model reduces to the HJM model up to a re-scaling.*

Recalling from [2], Baaquie, et al. gave the moment generating function of the quantum field theory by the Feynman path integral:

$$Z[J] = \frac{1}{Z}\int DAe^{\int_0^{t*} dt \int_t^{t+T_{FR}} dx J(t,x)A(t,x)e^{S[A]}} \tag{11}$$

Thus, with some changes of variables and subsequent calculations given in [2],

$$Z[J] = e^{\frac{1}{2}\int_0^{t_*} dt \int_t^{T_{FR}} d\theta\theta' J(t,\theta)D(\theta,\theta';t'T_{FR})J(t,\theta')} \tag{12}$$

for $\theta = x - t, \theta' = x' - t$, and the *propagator* $D(\theta, \theta'; t, T_{FR})$ given by

$$D(\theta,\theta' t, T_{FR}) = \frac{\mu}{sinh^3(\mu T_{FR})} \left\{ \begin{array}{c} sinh\mu(T_{FR}-\theta)sinh\mu\theta'\{1+sinh^2(\mu T_{FR})\Theta(\theta'-\theta)\}+ \\ sinh\mu(T_{FR}-\theta)sinh\mu\theta'\{1+sinh^2(\mu T_{FR})\Theta(\theta'-\theta)\}+ \\ cosh(\mu T_{FR})\{1+sinh\mu(T_{FR}-\Theta)sinh\mu(T_{FR}-\Theta')\} \end{array} \right\} \tag{13}$$

that is, the unconstrained boundary conditions, as discovered by Baaquie, et al. And, by the following:

Assumption 3.0.2. *Assuming the field at boundary $t = x$ (that is, $A(t,t)$) is distributed normally with the variance a.*

Since it is well known that short term interest rates are heavily influenced by central banks, the *propagator* becomes:

$$D_1(\theta,\theta') = D(\theta,\theta') - \frac{D(0,\theta)D(0,\theta')}{D(0,\theta')+a} \tag{14}$$

Remark 3.0.2. *Thus, it is obvious that any of the results due to the no arbitrage condition is not affected by the mean of the field at the boundary.*

In addition, from the *propagator* $D(\theta, \theta'; t, T_{FR})$, also noted Baaquie, et al., the *correlator* of the field $A(t,\theta)$, is given by

$$E(A(t,\theta)A(t,\theta')) = \delta(t-t')D(\theta,\theta';t,T_{FR}) \tag{15}$$

Thus, it can be readily shown that the no arbitrage condition is satisfied only when

$$\alpha(t,x) = \sigma(t,x)\int_t^x dx' D(s,x';t,T_{FR})\sigma(t,x') \tag{16}$$

And again, in the limit $\mu \longrightarrow 0$, $D \longrightarrow 1$, the one-factor HJM model is obtained as:

$$\alpha(t,x) = \sigma(t,x)\int_t^x dx'\sigma(t,x') \tag{17}$$

4 Empirical Experimentation

The empirical experimentation with the quantum-theoretic model of HJM with Baaquie, et al.

⇒ Uses daily closing prices for eurodollar futures prices as a measure of the forward rates.

⇒ Linearly interpolates the eurodollar futures prices, covering 846 days over the 1990s, to calculate forward rates at 3 month intervals (taken to be a good approximation to the instantaneous forward rate) following from [12], such that the dataset spanned 846 trading days covering the 1990s with forward rates 7 years into the future available.

⇒ Parameterizes the forward rates as $f(t,\theta)$, rather than $f(t,x)$ to simplify analysis considerably (since the domain shape in the (t,σ) variables is rectangular) and the focus is on the main quantities:

- $$V(\theta) = \sqrt{<\delta f^2(t,\theta)>} \tag{18}$$

- $$C(\theta) = \frac{< \delta f(t, \theta_{min})(\delta f(t, \theta) - \delta f(t, \theta_{min})) >}{< \delta f^2(t, \theta_{min}) >} \tag{19}$$
- $$r(\theta) = \frac{V(\theta)}{C(\theta) + 1} \tag{20}$$

(again in line with [17] *such that differences taken over one trading day* (ϵ)*,* $\delta f(t, \theta) = f(t + \epsilon, \theta) - f(t, \theta)$ *for* θ_{min} *three months – assuming there are 250 trading days in a year – to obtain the discretization* $\theta(0) = \frac{1}{\epsilon}$*).*

In addition,

By the one-factor HJM model, expressions for the above quantities, accurate to zeroth order in ϵ, are derived as follows:

- $$V_{HJM}(\theta) = \sigma(\theta)\sqrt{\epsilon} \tag{21}$$
- $$C_{HJM}(\theta) = \frac{\sigma(\theta)}{\sigma(\theta_{min})} - 1 \tag{22}$$
- $$r_{HJM}(\theta) = \sigma(\theta_{min})\sqrt{\epsilon} \tag{23}$$

Thus, discretizing the Brownian motion process W as $W(t) = \sqrt{\frac{1}{\epsilon}}x$ (where x is a random number with the standard normal distribution), noting in particular that the ratio $r_{HJM}(\theta)$ is *independent* of $\sigma(\theta)$ and is in fact *constant.*

However, as stated by Baaquie, et al. the ratio calculated from the data was far from constant, confirming that the time translation invariant one-factpr HJM model remains inconsistent with the real evolution of the FRC (Forward Rate Curve) for any choice of function $\sigma(\theta)$ as expected.

Again deriving expressions for the above quantity to zeroth-order accuracy in ϵ using the unconstrained quantum field-theoretic model, we have the following:

- $$V_{QFT}(\theta) = \sigma(\theta)\sqrt{D(\theta, \theta; t, T_{FR})\epsilon} \tag{24}$$
- $$C_{QFT}(\theta) = \frac{\sigma(\theta)D(\theta, \theta_{min}; t, T_{TR})}{\sigma(\theta_{min})D(\theta_{min}, \theta_{min}; t, T_{FR})} - 1 \tag{25}$$

Thus, giving the ration $r(\theta)$ in this model to be:

- $$r_{QFT}(\theta) = \frac{\sigma(\theta_{min})\sqrt{\epsilon D(\theta, \theta; t, T_{FR})}D(\theta_{min}, \theta_{min}; t, T_{FR})}{D(\theta, \theta_{min}; t, T_{TR})} \tag{26}$$

which Baaquie, et al. noted to be no longer constant.

However still independent of $\sigma(\theta)$, this suddenly no-longer-constant ratio has the possibility of fitting the ratio in order to find the μ and $\sigma(\theta_{min})$.

Thus, Baaquie, et al. taking the limit of $T_{FR} \to \infty$ (as required) and using the Levenberg-Marquardt method [13] obtained the non-linear least squares fit (as shown in table I of his paper [1]), with confidence intervals obtained through the bootstrap method [14] and an alternative confidence interval by dividing the data into series of 500 days starting from the first day, second day,... calculating function $r(\sigma)$, and fitting parameters (for the resulting 346 data sets as shown in figure 1 of [1]). And, similar to estimates of $\sigma(\theta)$ (plotted in figure 3 of [1]) for the one-factor HJM model, two different estimates (plotted in figure 2 of [1]) of the function $\sigma(\theta)$ are obtained using equations 22 and 23.

Remark 4.0.1. *Thus, the HJM model was shown to be inconsistent with the data, while on the contrary, the quantum field-theoretic model was consistent with data. Besides, constant or exponential forms, commonly used in the literature, was very far from the volatility function for the HJM model derived from the data.*

Also, Baaquie, et al. repeated the same procedure for the constrained quantum field-theoretic model, and showed that the agreement between the two functions is better than in the case of the unconstrained model – as may be expected due to the additional parameter involve – based on the obtained results depicted in table II, the fitted ratio shown in figure 4 of [1], and the two estimates of $\sigma(\theta)$ presented in figure 5 of [1], although the model may be over-specified since different values of the parameters give rise to very similar values for $r(\theta)$ as reflected by the large confidence intervals.

Furthermore, based on the assumption

Assumption 4.0.1. • *that σ is only a function of σ, and α is also only a function of θ, and*
• *that the initial FRC is flat or that the effect of the initial FRC becomes negligible after a long time*

the mean spread between the forward rates and the spot rate, given by:

$$s(\theta) = < f(t,\theta) - f(t,\theta_{min}) > \tag{27}$$

which is essentially a *linear sum of two parts in the model: the market price of risk and the no arbitrage condition*)

is derived thus as:

$$S_{QFT} = (\theta - \theta_{min}) \lim_{t\to\infty} \alpha(t) - \int_{\theta_{min}}^{\theta} \alpha(t)dt \tag{28}$$

(for the no arbitrage condition in the quantum field-theoretical model)

where

$$\alpha_{QFT}(t) = \sigma(t) \int_0^t \sigma(\theta) D(t,\theta;t,T_{FR}) d\theta \tag{29}$$

Baaquie, et al. calculated the spread *(due to the no arbitrage condition),* by applying numerical integration by trapezoidal method (chosen due to the relative inaccuracy in the estimation of $\sigma(\theta)$ in the first place) to one of the estimates of $\sigma(\theta)$ – either one giving similar results – and observed that the calculated spread was significantly smaller than the actual spread, even for when the procedure is repeated for the constrained quantum field-theoretic model.

Remark 4.0.2. *Thus, giving showing consistency with the existence of the spread due to risk aversion (although a significant portion of the spread could be derived from the way the forward rate curve evolves).*

5 Conclusion

Classical (stochastic) models of forward rates have been shown to be inadequate to fit for real phenomena happening with justifiable dyamical quantum causes. However, the quantum field theory (QFT) method, which is still undergoing development has been shown to hold much nicer features, even for rather fundamental models such the binomial model for forward rates. In addition, the QFT:

• Presents a new way to test models such as the one-factor, time translation invariant Heath-Jarrow-Morton, Baaquies one-factor, and the time translation invariant quantum field-theoretic model

• Shows relatively higher consistency with data (even when the boundary conditions are constrained so that it may reflect the special nature of the spot rate, and the parameters can not be sufficiently and accurately derived using the method), and

• Explains a significant portion of the spread between the forward rates and spot rate better than the classical models.

References

[1] Baaquie, B.E., Srikant, M An Empirical Investigation of a Quantum Field Theory of Forward Rates, *Physical Review E*, 64 016121-1 to 016121-16, 2001.

[2] Baaquie, B.E. Quantum Field Theory of Treasury Bonds, *Intl. Workshop on Nonlinear Physics, Theory and Experiment II*, Gallipoli, Lecce, 2002.

[3] Baaquie, B.E., Coriano, C., and Srikant, M. Quantum Mechanics, Path Integrals and Option Pricing: Reducing the Complexity of Finance, *Intl. Workshop on Nonlinear Physics, Theory and Experiment II*, Gallipoli, Lecce, 2002.

[4] Björk, T. *Arbitrage theory in continuous time*, Oxford University Press, 1998.

[5] Vasicek, O. An equilibrium characterization of the term structure, *Journal of Financial Economics*, 5, 1977.

[6] Cox, J., Ingersoll, J., and Ross, S. An inter-temporal general equilibrium model of asset prices, *Econometrica*, 53, 2, 1985a.

[7] Cox, J., Ingersoll, J., Ross, S. An inter-temporal general equilibrium model of asset prices, *Econometrica*, 53, 2, 1985b.

[8] Hull, J., and White, A. Pricing interest rate derivative securities, *The Review of Financial Studies*, 3, 1, 1990.

[9] Bjork, T., and Hyll, M. On the inversion of the yield curve, *Working paper*, Stockholm School of Economics, 2000.

[10] Heath, D., Jarrow, R., and Morton, A. Bond pricing and the term structure of interest rates: a new methodology, *Econometrica*, 60, 1, 1992.

[11] Brown, R.G., and Schaefer, S.M. Interest rate volatility and the shape of the term structure, *Phil. Trans. R. Soc.*, Lond., 1994.

[12] Bouchaud, J.P., Cont R., El-Karoui N., Potters M., and Sagna N. Phenomenology of the Interest Rate Curve, *Applied Mathematical Finance*, 6, 1997.

[13] Marquardt, D.W. An algorithm for least-squares estimation of nonlinear parameters, *Journal of the Society for Industrial and Applied Mathematics*, 11, 1963.

[14] Efron, B. The jacknife, the bootstrap, and other resampling plans, *SIAM*, Philadelphia, 1982.

[15] Buhler W., Uhrig-Homburg M., Walter U., and Weber T. An Empirical Comparison of Forward-Rate and Spot-Rate Models for Valuing Interest-Rate Options, *Journal of Finance*, 54, 1999.

[16] Flesker B., Testing of the Heath-Jarrow-Morton / Ho-Lee Model of Interest Rate Contingent Claims Pricing, *Journal of Financial and Quantiative Analysis*, 38, 1993.

[17] Sim, A.B., and Thurston, D.C. An empirical study of a new class of no-arbitrage-based discrete models of the term structure, *The Journal of Financial Research*, 19, 1996.