Math H104: Honors Introduction to Analysis **Fall 2005**

Lecture 1: August 30

Lecturer: Yuval Peres *Scribe: Thomson Nguyen*

Let $\mathbb{Z}$ be the set of all integers $\{0, 1, -1, 2, -2, \ldots\}$ and let $\mathbb{N}$ be the positive integers $\{1, 2, 3, \ldots\}$. Denote the rational numbers by $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\}$. The ancient Greeks already discovered that rational numbers are not sufficient to describe certain natural geometrical quantities, such as the diagonal in a square of side 1.

Proposition 0.1. $\sqrt{2} \notin \mathbb{Q}$. *That is, for every $a, b \in \mathbb{Z}$ with $b \neq 0$, we have $(a/b)^2 \neq 2$.*

Proof. Suppose $(a/b)^2 = 2$ with $a, b \in \mathbb{Z}$. We may assume that $a, b > 0$, otherwise we replace a, b by their absolute values. We also may assume that we chose a solution with a minimal. The equation $a^2 = 2b^2$ implies that a is even, and therefore a^2 is divisible by 4. Consequently $b^2 = a^2/2$ is even whence b is even. Therefore we can replace a and b by $a/2$ and $b/2$, and obtain a smaller pair of integers where the ratio of their squares is 2. This contradicts the minimality of a. □

The construction of the real numbers $\mathbb{R}$ can be done either via Dedekind cuts, or using Cauchy sequences. A **Dedekind cut** $A|B$ consists of a pair of disjoint nonempty sets $A, B \subset \mathbb{Q}$, such that $A \cup B = \mathbb{Q}$ and $a < b$ holds for all $a \in A$ and $b \in B$. We also require that A has no largest element.

A pertinent example of a Dedekind cut is $A|B$ where

(1) $$A = \{x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2\} \text{ and } B = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 2\}.$$

We will return to Dedekind cuts later.

Recall that a sequence $\{x_n\}$ **converges** to a limit L (in symbols, $x_n \to L$ as $n \to \infty$) if for any $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that $|x_n - L| < \epsilon$ for all $n > n_0$. For now, focus on $x_n, L, \epsilon \in \mathbb{Q}$. This also applies to the next definition. However, these definitions will apply more generally later. We need a more sophisticated definition that describes when the members of a sequence are getting closer to each other without refering to any limit.

Definition 0.2. *A sequence $\{x_n\}_{n=1}^{\infty}$ is a* ***Cauchy sequence*** *if for all (rational) $\epsilon > 0$, there exists an N such that $m, n > N \Rightarrow |x_m - x_n| < \epsilon$.*

For example, the sequence $\{3.1, 3.14, 3.141.3.1415, 3.14159, ...\}$ where each time we add another digit in the decimal expansion of π, is a Cauchy sequence. As we shall see later in the course, $\pi \notin \mathbb{Q}$, so this sequence does not converge in $\mathbb{Q}$. Similarly, if $x_n^2 \to 2$, then $\{x_n\}$ cannot converge to any rational L.

Problem 0.3 (Challenge)**.** *Find an explicit sequence $\{x_n\} \subset \mathbb{Q}$ such that $x_n^2 \to 2$ for all $x_n > 0$.*

Following the preceding example, we can take $x_1 = 1.4$, and $x_n = x_{n-1} + \frac{a_n}{10^n}$ for $n > 1$, where a_n is the largest integer a such that $(x_{n-1} + \frac{a}{10^n})^2 < 2$. Then $\{x_n\}$ is a Cauchy sequence, and $x_n^2 \to 2$ as $n \to \infty$.

Here is an idea for a more insightful solution, motivated by a standard algorithm to approximate square roots. Let

$$x_1 = 2 \text{ and } x_n = \frac{1}{2}\Big(x_{n-1} + \frac{2}{x_{n-1}}\Big) \text{ for } n > 1\,. \tag{2}$$

By induction, $x_n \in \mathbb{Q}$ for all n.

Problem 0.4 (Exercise)**.** *For the sequence in (2), check that $x_{n+1} < x_n$ for all $n > 0$ and that the Cauchy property holds. Hint: Consider $x_n^2 - 2$.*

To ensure that a sequence $\{y_n\}$ is Cauchy, it is **not** enough to verify that $y_n - y_{n-1} \to 0$ as $n \to \infty$.

Example 0.5. *Consider $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, so that $H_n - H_{n-1} = \frac{1}{n} \to 0$. Nevertheless, $\{H_n\}$ is not a Cauchy sequence. To see this, take $\epsilon = 1/3$, for instance. Given any N, we must find $m, n > N$ with $|H_n - H_m| \geq 1/3$. Let $m = N + 1$ and $n = 2m$. Then*

$$H_{2m} - H_m = \frac{1}{m+1} + \frac{1}{m+2} + \ldots + \frac{1}{2m} \geq \frac{m}{2m} = \frac{1}{2}.$$

We are done.

In the preceding example, the sequence H_n is not bounded.

Problem 0.6 (Exercise)**.**
- *Show that every Cauchy sequence is bounded.*
- *Show that every convergent sequence is a Cauchy sequence.*
- *Find an example of a bounded sequence $\{y_n\}$ such that $y_n - y_{n-1} \to 0$ yet $\{y_n\}$ is not a Cauchy sequence. Hint: Consider the distance from H_n to the nearest integer.*

To define real numbers via Cauchy sequences, we must deal with the fact that many different sequences might converge to the same limit.

Definition 0.7. *Suppose $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences of rational numbers. We say that $\{x_n\}$ is* ***equivalent to*** *$\{y_n\}$, and write $\{x_n\} \sim \{y_n\}$, if $x_n - y_n \to 0$.*

Given a Cauchy sequence $\{x_n\} \subset \mathbb{Q}$, consider its **equivalence class**

$$\overline{\{x_n\}} = \{\text{all sequences } \{y_n\} \text{ such that } \{x_n\} \sim \{y_n\}\}\,.$$

We can **define a real number** as such an equivalence class. To do so, and still think of $\mathbb{Q}$ as a subset of $\mathbb{R}$, we identify every rational number with the equivalence class of (Cauchy) sequences converging to it.

Math H104: Honors Introduction to Analysis **Fall 2005**

Lecture 2: September 01

Lecturer: Yuval Peres *Scribe: Orie Shelef*

1. Primes

Question: Show that there are infinitely many primes: 2,3,5,7,...
Proof (Euclid): Given any finite set of primes $p_1, \dots, p_k$, we construct another one. Consider $N = p_1 p_2 ... p_k + 1$. This N must have some prime factor q (possibly $q = N$). Since $N - 1$ and N cannot both be divisible by q, it follows that q is different from $p_1, \dots p_k$.

Next, let $P_1, P_2, P_3, \dots$ be the ordered list of all primes:
Theorem (Euler) $\sum_{j=1}^{\infty} \frac{1}{P_j} = \infty$ We'll prove this later.
This theorem shows that the sequence of all primes p_j "does not grow too fast".
Amusing fact: $\sum \frac{1}{P} < 5$ where the sum is over all "*known primes*", that is those primes that have ever been identified.

2. Construction of Real Numbers

Recall that a Dedekind Cut $A|B$ satisfies the following conditions:

- $A \bigcup B = \mathbb{Q}$;
- $A \bigcap B = \emptyset$;
- $A \neq \oslash$, $B \neq \oslash$;
- if $a \in A$ and $b \in B$ then $a < b$;
- A has no largest element: $\forall a \in A$, $\exists a_1 \in A : a_1 > a$.

There exist two types of cuts depending on whether B has a smallest element (Type 1) or not (Type 2).
Examples:

- Type 1: $A = \{x \in \mathbb{Q} : x < 3\}$ and $B = \mathbb{Q} \setminus A = \{x \in \mathbb{Q} : x \geq 3\}$;
- Type 2: $A = \{x \in \mathbb{Q} : x^2 < 2 \text{ or } x < 0\}$ and $B = \mathbb{Q} \setminus A$.

Type 1 cuts correspond to rational numbers. For any $q \in \mathbb{Q}$ we have a type 1 cut $A_q|B_q$ where $A_q = \{x \in \mathbb{Q} : x < q\}$ and $B_q = \{x \in \mathbb{Q} : x \geq q\}$; conversely, any type 1 cut can be represented this way.

We can now define the sum of two cuts:

$$(A_1|B_1) + (A_2|B_2) = (A_1 + A_2|\mathbb{Q} \setminus (A_1 + A_2))$$

where set operations are defined as $A_1 + A_2 = \{a_1 + a_2 | a_1 \in A_1, a_2 \in A_2\}$.

3. Supremum and Infimum

Definition 3.1. $(A_1|B_1) < (A_2|B_2)$ *iff* $A_1 \subset A_2$ *and* $A_1 \neq A_2$.

Fact: (Check!) For any two distinct cuts $(A_1|B_1)$ and $(A_2|B_2)$, we have $(A_1|B_1) < (A_2|B_2)$ or $(A_2|B_2) < (A_1|B_1)$, but not both.

If S is a set (in $\mathbb{Q}$ or in $\mathbb{R}$) and $x \in \mathbb{R}$ (or $\mathbb{Q}$) we say that *x is an upper bound for s* if $s \leq x$ for all $s \in S$. We say that $x_0 \in \mathbb{R}$ is the **least upper bound** of S, and write $x_0 = \sup S$, if x_0 is an an upper bound for S, and for each upper bound x of S we have $x \geq x_0$. If S has no upper bound then we write $\sup S = \infty$.

Similarly, define $\inf S = y_0$ if y_0 is a lower bound for S and any lower bound y for S satisfies $y \leq y_0$. if S has no lower bound then let $\inf S = -\infty$.

Note: $S = \{x \in \mathbb{Q} : x^2 < 2\}$ has no supremum in $\mathbb{Q}$. Why? Given an upper bound $z \in \mathbb{Q}$ for S, we can always find a smaller upper bound. Given $z^2 > 2$, we seek $z_1 \in \mathbb{Q}$ such that $0 < z_1 < z$ and $z_1^2 > 2$.

- *One suggestion*: Consider $x_k = z - 2^{-k}$ for $k \in \mathbb{N}$. These x_k are all rational, positive, and $x_k < z$ for all k. Since $z^2 > 2$ and $x_k \to z$ we have $x_k^2 > 2$ for some k. That x_k will be our z_1.
- *A nicer suggestion*: Take $z_1 = (z + 2/z)/2$. Clearly $0 < z_1 < z$. We must check that $z_1^2 > 2$. Indeed $z_1^2 = (z^2 + 4/z^2 + 4)/4$. This is strictly greater than 2 iff $z^2 + 4/z^2 > 4$, which is true since the left hand side minus the right hand side can be written as $(z - 2/z)^2 > 0$.

4. Key Property of $\mathbb{R}$

Proposition 4.1. *If* $S \in \mathbb{R}$ *has an upper bound then* $\exists \sup S \in \mathbb{R}$.

Idea of Proof using Cuts: S is a collection of cuts. Let $A_* = \bigcup_{(A|B) \in S} A$ and $B_* = \bigcap_{(A|B) \in S} B$. Then $(A_*|B_*) = \sup S$. Check that indeed, $(A_*|B_*)$ is a cut and satisfies the definition of supremum.

5. Homework

Due Thursday, September 8th. From Book: 9, 15, 16(a)(b)(c) (Pages 41-44) And the following:

Suppose $x \in \mathbb{Q}$ solves $x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ with $a_i \in \mathbb{Z}$. Show $x \in \mathbb{Z}$. *Hint:* This is an example of a monic polynomial (leading coefficient is 1) with integer coefficients. such polynomials have only integer or irrational solutions. Try simpler monic polynomials first. $x + a_0 = 0$ is too easy. $x^2 + a_1x + a_0 = 0$ can be solved using the quadratic formula. Then show that solutions of this equation are integer or irrational without the quadratic formula and apply that method to the original question.

Math H104: Honors Introduction to Analysis **Fall 2005**

Lecture 3: September 06

Lecturer: Yuval Peres *Scribe: Cordelia Csar*

Recursion

In Lecture 1, we discussed the recursion

$$x_1 = 2, x_n = \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right) \text{ for } n > 1$$

We claim that $x_k > x_{k+1}$ for all $k \geq 1$ and $x_n^2 \to 2$. **Why?** Let $y_n = x_n^2 - 2$, which means $y_2 = \frac{1}{4}$. For all $n > 1$, we have

$$y_{n+1} = x_{n+1}^2 - 2 = \frac{1}{4}\left(x_n + \frac{2}{x_n}\right)^2 - 2 = \frac{y_n^2}{4x_n^2}\,.$$

By induction, $1 \leq x_n \leq 2$ for all n. Therefore, $y_n \leq 2$, whence $0 < y_{n+1} \leq \frac{y_n^2}{4} < y_n$. In particular, $y_n \leq \frac{1}{4}$ for all $n \geq 2$. Furthermore, $0 < y_{n+1} \leq \frac{y_n}{16}$ for $n > 1$. From above,

$$\begin{aligned} x_{n+1}^2 &< x_n^2 \\ x_{n+1} &< x_n \quad \text{for } n > 1 \end{aligned}$$

Next note that $\{x_n^2\}$ is a Cauchy sequence, as any convergent sequence is a Cauchy sequence. Convergence $z_n \to L$ means that for all ϵ there exists n_0 such that, for all $n > n_0$, $|z_n - L| < \epsilon$. Given ϵ, we can check that $\{z_n\}$ satisfies the Cauchy criterion by finding n_0 such that $|z_n - L| < \frac{\epsilon}{2}$ for all $n > n_0$. This implies that, for all $n, m > n_0$, we have $|z_n - z_m| \leq |z_n - L| + |z_m - L| < \epsilon$. Our sequence $\{x_n\}$ satisfies $x_n \geq 1$ so $|x_n - x_m| = \dfrac{|x_n^2 - x_m^2|}{x_n + x_m} \leq |x_n^2 - x_m^2|$, which implies that $\{x_n\}$ is a Cauchy sequence.

Real Numbers

Definition 5.1. *A set $\widetilde{\mathbb{R}}$ can be identified with the real numbers if it is totally ordered by "$<$" and contains (a copy of) $\mathbb{Q}$ with its order and*

$$\{\textit{Dedekind cuts in } \mathbb{Q}\} = \{\{x \in \mathbb{Q} : x < r\} | \{x \in \mathbb{Q} : x \geq r\}\} \textit{ for } r \in \widetilde{\mathbb{R}}$$

Question 5.2. *Why does $\widetilde{\mathbb{R}}$, defined as equivalence classes of Cauchy sequences, satisfy this?*

Suppose $r = \overline{\{x_n\}} \in \widetilde{\mathbb{R}}$. Then define

$$A_r = \{q \in \mathbb{Q} : \text{ there exists } q_1 > q \text{ such that} q_1 \leq x_n \text{ for all but finitely many } n\}$$

and $B_r = \mathbb{Q} \setminus A_r$. To check that A_r and B_r are well defined, (i.e., they depend only on r and not on the chosen representative $\{x_n\}$) we need to verify the following.

Proposition 5.3. *Let $\{x_n\}$ and $\{y_n\}$ be equivalent Cauchy sequences in $\mathbb{Q}$. Then there exists $q_1 > q$ such that $q_1 < x_n$ for all but finitely many n,* ***if and only if*** *there exists $q_2 > q$ such that $q_2 < y_n$ for all but finitely many n.*

Proof. $\Rightarrow$ Given q_1 with $q_1 < x_n$ for all but finitely many n, take $q_2 = \dfrac{q+q_1}{2} < q_1$. We wish to show that $y_n > q_2$ for all but finitely many n. Since $\{x_n\} \sim \{y_n\}$, we have $|y_n - x_n| < \dfrac{q_1 - q}{2}$ for all but finitely many n. The $\Leftarrow$ argument is similar. $\square$

To complete the equivalence, we need to, given a Dedekind cut $A|B$, construct an element $r \in \widetilde{\mathbb{R}}$. We can do this in the following manner. Let $x_1 =$ largest element of $\dfrac{\mathbb{Z}}{10}$ in A, where

$$\frac{\mathbb{Z}}{10} = \{\ldots, -.3, -.2, -.1, 0, .1, .2, .3, \ldots\}.$$

More generally, let $x_n =$ largest element of $\dfrac{\mathbb{Z}}{10^n}$ in A. This keeps adding one addition digit of precision. Finally, if r is the equivalence class of $\{x_n\}$, one can check that $A_r = A$ and $B_r = B$.

Math H104: Honors Introduction to Analysis **Fall 2005**

Lecture 4: September 8

Lecturer: Yuval Peres *Scribe: Stephen Bianchi*

Chapter 1, Exercise 9(a)
Let $A|B$ and $\tilde{A}|\tilde{B}$ be cuts in $\mathbb{Q}$. We defined cut addition as

$$A|B + \tilde{A}|\tilde{B} = (A + \tilde{A})|(\mathbb{Q} \setminus (A + \tilde{A})).$$

We do this because if we were to define cut addition as

$$A|B + \tilde{A}|\tilde{B} = (A + \tilde{A})|(B + \tilde{B}),$$

then we would not necessarily get a cut on the right hand side. For example, consider the following cuts in $\mathbb{Q}$,

$$A = \{x < \sqrt{2}\},\ B = \mathbb{Q} \setminus A$$
$$\tilde{A} = \{x < 5 - \sqrt{2}\},\ \tilde{B} = \mathbb{Q} \setminus \tilde{A}.$$

Then,

$$A + \tilde{A} = \{x \in \mathbb{Q} : x < 5\}$$
$$B + \tilde{B} = \{\tilde{x} \in \mathbb{Q} : \tilde{x} > 5\}$$

but

$$5 \notin (A + \tilde{A}) \cup (B + \tilde{B}).$$

Question
Find $x, y \in \mathbb{R} \setminus \mathbb{Q}$ such that $x^y \in \mathbb{Q}$, or at least show that such and x and y exist.

One explicit solution is

$$e^{\log 2} = 2.$$

But how do we know e and $\log 2$ are irrational?

General solution:
First try $z = \sqrt{2}^{\sqrt{2}}$. If $z \in \mathbb{Q}$ then we are done. If $z \notin \mathbb{Q}$ then consider

$$z^{\sqrt{2}} = (\sqrt{2})^{\sqrt{2}\cdot\sqrt{2}} = 2.$$

So either $z \in \mathbb{Q}$ to begin with, or taking $z^{\sqrt{2}}$ gives the solution. As it turns out, $z \notin \mathbb{Q}$, which is a special case of the *Gelfond-Schneider* theorem.

Going back to our explicit solution, we need to show that e and $\log 2$ are irrational. To show that e is irrational, we start with the following,

$$\begin{aligned} e^{-1} &= 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots \\ e &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots \end{aligned}$$

Now suppose $e = \frac{p}{q}$ $(p, q \in \mathbb{N})$. Then $e^{-1} = \frac{q}{p}$, and $p!e^{-1} \in \mathbb{Z}$.
If p is odd, then

$$p!e^{-1} = m + \frac{p!}{(p+1)!} - \frac{p!}{(p+2)!} + \cdots$$

If p is even, then

$$p!e^{-1} = m - \frac{p!}{(p+1)!} + \frac{p!}{(p+2)!} - \cdots$$

Where $m \in \mathbb{Z}$. The sum of the alternating series above is between m and $m+1$. That is,

$$m < p!e^{-1} < m + \frac{p!}{(p+1)!} = m + \frac{1}{p+1} < m + 1.$$

Hence, $m < p!e^{-1} < m+1$, and $p!e^{-1}$ can not be in $\mathbb{Z}$.

Exercise
A warm-up exercise for showing $\log 2 \notin \mathbb{Q}$ is to show that $\log_3 2 \notin \mathbb{Q}$.

Cauchy Sequences
We have discussed the least upper bound property of $\mathbb{R}$. Next, we want to use this to show that Cauchy sequences in $\mathbb{R}$ converge (to a limit $\in \mathbb{R}$).

Step 1: Any Cauchy sequence $\{x_n\} \in \mathbb{R}$ is bounded.

Proof. Let $\epsilon = 1$ in the definition of Cauchy sequence. Then

$$\exists N : n, m > N \Longrightarrow |x_m - x_n| < 1$$

If we take $\{x_1, x_2, \ldots, x_N\} \cup [x_{N+1} - 1, x_{N+1} + 1]$, then this set is bounded. Let's say it is bounded in the interval $[-M, M]$. Then the sequence $\{x_j\}_{j=1}^{\infty} \subset [-M, M]$. □

Step 2: Any monotone, increasing, bounded sequence ($a_1 < a_2 < \ldots < a_n \leq M$, for all n) converges. Take $L = \sup\{a_j\}_{j=1}^{\infty}$, we claim $a_n \longrightarrow L$.

Proof. Given $\epsilon > 0$, we know $L - \epsilon$ is not an upper bound for $\{a_j\}$. This implies

$$\exists k : n > k \Longrightarrow a_n > L - \epsilon \Longrightarrow L - \epsilon < a_n \leq L.$$

Since our choice of ϵ was arbitrary, we conclude that $a_n \longrightarrow L$. □

Theorem 5.4. *Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$, then $\{x_n\}$ converges.*

Proof. We want to choose a monotone subsequence of $\{x_n\}$. One approach is to let $a_n = \max\{x_j\}_{j=1}^{n}$ (i.e., let a_n be the largest element in the firsts n elements). Then $a_n \longrightarrow L$. Note, however, that this approach can fail! Consider $x_k = \frac{1}{k}$, then $x_k \longrightarrow 0$ and $a_n = 1$, for all n. So in this case the limit of a_n and that of x_n are totally different. So we must do better.

Consider $b_k = \sup\{x_n : n \geq k\}$. The sequence $\{b_k\}$ is bounded. Also, $b_{k+1} \leq b_k$ (since b_{k+1} is the supremum of a smaller set); b_k is an upper bound for $\{x_n : n \geq k+1\}$, but b_{k+1} is the least upper bound, so $b_{k+1} \leq b_k$. Since $\{-b_k\}$ converges, this implies that $\{b_k\}$ converges to some limit, call it L_1. Now we want to check that $\{x_j\} \longrightarrow L_1$. Given $\epsilon > 0$,

$$\exists j : k > j \Longrightarrow |b_k - L_1| < \epsilon$$

$$\exists N : n, m > N \Longrightarrow |x_m - x_n| < \epsilon$$

Take $N_1 = \max\{N, j\}$. Then for all $k > N_1$ we have $|b_k - L_1| < \epsilon$, and there is an $n \geq k$ such that $b_k - \epsilon \leq x_n \leq b_k$. Finally, for all $m > N_1$,

$$|x_m - L_1| \leq |x_m - x_n| + |x_n - b_k| + |b_k - L_1| < 3\epsilon.$$

□

Definition 5.5. *An infinite set A is called* **denumerable** *if it can be written as a sequence, $A = \{a_1, a_2, a_3, \ldots\}$ (i.e., $A = \{f(1), f(2), f(3), \ldots\}$). More formally, A is denumerable if there is a one-to-one, onto map $f : \mathbb{N} \longrightarrow A$.*

Definition 5.6. *A is* **countable** *if it is either finite or denumerable.*

For example, **Integers ($\mathbb{Z}$)** $\mathbb{Z}$ is denumerable.

Proof. Simply write $\mathbb{Z}$ as $\{0, -1, 1, -2, 2, \ldots\}$. □

Rational Numbers ($\mathbb{Q}$)
$\mathbb{Q}$ is denumerable.

Proof. To show this let's first check that $\mathbb{N} \times \mathbb{N} = \{(a, b) \in \mathbb{N}\}$ is denumerable (i.e., $|\mathbb{N} \times \mathbb{N}| = \aleph_0$). Observe that if $\{A_j\}_{j=1}^{\infty}$ are finite, then their union is countable. To see this, just write out the elements of each set in order. Hence, $|\cup_{j=1}^{\infty} A_j| = \aleph_0$. Then for $|\mathbb{N} \times \mathbb{N}|$ consider the sets,

$$A_j = \{(a, b) \in \mathbb{N} \times \mathbb{N} : a + b = j\}.$$

Finally, define $\mathbb{Q} = \cup_{j=1}^{\infty} Q_j$, where

$$Q_j = \left\{ \frac{a}{b} : \text{ where } \frac{a}{b} \text{ is reduced}, b \neq 0, b \in \mathbb{Z}, a \in \mathbb{N}, \text{ and } : a + |b| = j \right\}.$$

□

Real Numbers ($\mathbb{R}$) $\mathbb{R}$ is not countable. The proof by contradiction is due to *George Cantor* (circa 1870).

Proof. Suppose $\mathbb{R}$ could be enumerated in a sequence $\{x_1, x_2, x_3, \ldots\}$. Further suppose $\mathbb{R}$ is countable. Then $(0, 1)$ is a countable subset of $\mathbb{R}$, and can be enumerated as $\{x_1, x_2, x_3, \ldots\}$, where

$$x_j = \sum_{k=1}^{\infty} \frac{x_{j,k}}{10^k} = 0.x_{j,1}x_{j,2}x_{j,3}\ldots \text{ with } 0 \leq x_{j,k} \leq 9,$$

and our expansion does not terminate in an infinite sequence of 9s. Define

$$y = \sum_{k=1}^{\infty} \frac{y_k}{10^k}$$

where $y_k = 2$, if $x_{k,k} = 1$, and $y_k = 1$, if $x_{k,k} \neq 1$. Then

$$y = 0.2112111211221\ldots \quad \in (0, 1).$$

For $(0, 1)$ to be a countable subset of $\mathbb{R}$, y must appear (somewhere) in the sequence $\{x_j\}$ (i.e., there must be a j such that $y = x_j$). But $y_j \neq x_{j,j}$, so $y \neq x_j$. Thus y appears nowhere in our sequence. This contradicts our assumption that $(0, 1)$ is a countable subset of $\mathbb{R}$. □

Homework due 9/15

From Chapter 1 of the book: 33, 35, and 36(a). For 36(a), you need to know that for a polynomial of degree n, there are *at most* n roots in $\mathbb{R}$. Hint to see this: any polynomial can be written as

$$P(x) = Q(x)(x - a) + b.$$

And the following problem:

Given $x \in (0, 1)$, expand

$$x = \sum_{k=1}^{\infty} \frac{x_k}{10^k}$$

where the expansion does not terminate in 9s. Show that $x \in \mathbb{Q}$ if and only if this expansion is eventually periodic (example, $0.123432432432\ldots$).

Math H104: Honors Introduction to Analysis	**Fall 2005**

Lecture 5: September 13

Lecturer: Yuval Peres *Scribe: Thomson Nguyen*

Remark 5.7. *A polynomial f of degree n (over $\mathbb{R}$) has at most n real roots.*

Two proofs:

(1) Show by induction that any f of degree n can be written as $f(x) = (x-a)g(x) + 1$ for any fixed a.

(2)

$$\begin{aligned} f(x) &= f(x) - f(a) \\ &= \sum_{j=0}^{n} c_j(x^j - a^j) \\ &= \sum_{j=1}^{n} c_j(x-a)(x^{j-1} + ax^{j-2} + a^2x^{j-3} + \ldots + a^{j-2}x + a^{j-1}), \end{aligned}$$

where $f(x) = c_nx^n + c_{n-1}x^{n-1} + \ldots + c_1x + c_0$. Note that if $j = 1$, $x - a = (x-a) * 1$,; if $j = 2$, $x^2 - a^2 = (x-a)(x+a)$.

Sketch of the first proof. If $f(a) = 0$, then $x-a$ divides $f(x)$; that is, $f(x) = g(x)(x-a)$ where $g \in \mathbb{R}[x]$, $\mathbb{R}[x]$ is the set of polynomials over $\mathbb{R}$, and $\deg(g) = n-1$. We know this by induction on n. We may assume f is **monic**, or of the form $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0x^0$. □

Definition 5.8. *Given $y \in \mathbb{R}$, we say that*

(1) *y is an **algebraic** number if there exists $f \in \mathbb{Z}[x]$ such that $f(y) = 0$. [Equivalently, there exists $\tilde{f} \in \mathbb{Q}[x] with \tilde{f}(y) = 0$]*
(2) *y is a **transcendental** number if it is not algebraic.*
(3) *y is an **algebraic integer** if $f(y) = 0$ for some monic polynomial $f \in \mathbb{Z}[x]$.*

Remark 5.9. *Note that it was assigned as an exercise that the set $\mathbb{A}$ of algebraic numbers is countable. Thus, since we know $\mathbb{R}$ is uncountable, then we see that $\mathbb{A}$ is a strict subset of $\mathbb{R}$. The existence of transcendental numbers follows.*

Remark 5.10. *The set $\mathbb{A}$ is contained in the set $\mathbb{Z} \cup \{\mathbb{R} \backslash \mathbb{Q}\}$.*

Example 5.11. *$x = 0.1100010\ldots01\ldots$, with ones in the 1st, 2nd, 6th, 24th decimal place, and etc. i.e.,*

$$x = \sum_{k=1}^{\infty} 10^{-k!}$$

is transcendental.

Definition 5.12. *We say that two sets X, Y have the* ***same cardinality*** *if there is a $1-1$, onto mapping*[1] $f : X \to Y$. *[That is, for every $y \in Y$, there is a unique $x \in X$ with $f(x) = y$, we write $x = f^{-1}(y)$ and $f^{-1} : Y \to X$ which is 1-1 and onto.]*

Theorem 5.13 (Schroeder-Bernstein Theorem)**.** *Given sets X and Y, suppose there exist $1-1$ mappings $f : X \xrightarrow{1-1} Y$ and $g : Y \xrightarrow{1-1} X$. Then there exists a bijection $h : X \to Y$.*

Proof. Define

$$\begin{aligned} A_0 &= \{x \in X | x \neq g(y) \text{ for all } y \in Y\} \\ &= \{x \in X | x \notin g(Y)\}, \qquad \text{and} \\ B_0 &= \{y \in Y | y \notin f(X)\} \end{aligned}$$

Then set $A_1 = \{x \in X | x = g(y) \text{ for some } y \in B_0\}$, and $B_1 = \{y \in Y | y = f(x) \text{ with } x \in A_0\}$.

Inductively, define $A_k = \{x \in X | x = g(y) \text{ for some } y \in B_{k-1}\}$, and $B_k = \{y \in Y | y = f(x) \text{ for some } x \in A_{k-1}\}$. We then define $A_\infty = X \setminus \bigcup_{k=0}^{\infty} A_k = X \setminus \{A_0 \cup A_1 \cup A_2 \cup \ldots\}$. That is, $A_\infty = \{x|$ There is a sequence of preimages $x, y_1, x_1, y_2, x_2, y_3, x_3, \ldots\}$. Similarly set $B_\infty = Y \setminus \bigcup_{j=0}^{\infty} B_j$.

Let $h(x) = f(x)$ for $x \in A_\infty$, where $h : A_\infty \xrightarrow[onto]{1-1} B_\infty$.

$h(x) = f(x)$ for $x \in \bigcup_{i=0}^{\infty} A_{2j+1}$. □

Exercise 5.14. *Check that* $h : X \xrightarrow[onto]{1-1} Y$

Example 5.15. *The* ***ternary Cantor set*** *is* $\mathcal{C} = \left\{ \sum_{n=1}^{\infty} a_n 3^{-n} | a_n \in \{0, 2\} \text{ for all } n \right\}$. *We want to show that $\mathcal{C}$ and $[0, 1]$ have the same cardinality.*

To do so, we must find $1-1$ functions $f : \mathcal{C} \xrightarrow{1-1} [0, 1]$ and $g : [0, 1] \xrightarrow{1-1} \mathcal{C}$. We see immediately that one such f is f_{id}, or $f(x) = x$. For g, we can take the binary expansion of an $x \in [0, 1] : x = 0.x_1x_2x_3\ldots$, which is unique if we do not allow a terminating sequence of ones. Then $x = \sum_{k=1}^{\infty} \frac{x_k}{2^k}$ with $x_k \in [0, 1]$, but infinitely many x_k are zero. Then set $g(x) = \sum_{k=1}^{\infty} \frac{2x_k}{3^k}$. Since $f : \mathcal{C} \to [0, 1]$ and $g : [0, 1] \to \mathcal{C}$ are injective functions, we know that there exists a bijection by the Schroeder-Bernstein Theorem, and so the cardinalities are equal.

---

[1] A $1-1$ onto function can also be called a **bijective** function, or is a **bijection** between X and Y

Definition 5.16. *Two sets X, Y have the same cardinality if there exists a bijective function between the two.* $|X| < |Y|$ *if there exists an* $1-1$ *mapping* $f : X \to Y$. *Moreover,* $|X| > |Y|$ *if there exists an onto mapping* $g : X \to Y$.

Math H104: Honors Introduction to Analysis	**Fall 2005**

Lecture 6: September 15

Lecturer: Yuval Peres *Scribe: Matthew Bernard*

5.1. **Hilbert's Description of Cantor's Idea.** Suppose you are a inn-keeper at a hotel with an infinite, denumerable, set of rooms, numbered $\{1, 2, 3, ...\}$. The hotel is full, and then a new guest arrives. It's possible to fit the extra guest in by asking the guest who was in room k to move to room $k+1$ for all $k \geq 1$. Similarly, if an infinite sequence of new guests arrives, we can fit them all in by asking the occupant of room k to move to room $2k$ for all $k \geq 1$, and using the odd-numbered rooms that have all been vacated for the new arrivals.

5.2. **Metric Spaces.** A metric space (X, d) consists of a nonempty set X, and d, the distance function (also known as the metric) is a function $d : X \times X \to [0, \infty)$ satisfying the following three properties:

(1) $d(x, y) = 0 \Leftrightarrow x = y$ (d separates points)
(2) $d(x, y) = d(y, x)$, $\forall x, y \in X$ (Symmetry)
(3) $d(x, y) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in X$ (Triangle Inequality)

. Metric spaces are useful in many parts of pure Mathematics, as well as in applications to Computer Science and Biology. A very simple example of a metric space is to take any set X and define $d(x, y) = 1$ for $\forall x \neq y$ in X. Our most important example is: $X = \mathbb{R}^n$ with the Euclidean distance $d(x, y) = \|x - y\|_2 = (\sum(x_i - y_i)^2)^{\frac{1}{2}}$. These satisfy properties 1,2 of a metric. However, the triangle inequality is not obvious, so we shall prove it subsequently. The case $n = 1$ is already known.

Triangle Inequality: This uses the Cauchy-Schwarz inequality in $\mathbb{R}^n$: $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$ where $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ and and $\|x\|_2 = \sqrt{\langle x, x\rangle}$

Cauchy-Schwartz: We can assume that $x, y \neq 0$
First we consider the special case when $\|x\| = \|y\| = 1$. Then,

$$\begin{aligned} 0 &\leq \langle x + y, x + y\rangle \\ &= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \\ &= 2(1 + \langle x, y\rangle). \end{aligned}$$

Why, because $(x_i + y_i)(x_i + y_i) = x_i^2 + 2x_i y_i + y_i^2$. But $\| - y\| = 1$, Therefore $\langle x, y\rangle \geq -1$
$\Rightarrow -1 \leq \langle x, -y\rangle \leq (-1)(-1) = 1$
$\Rightarrow | < x, y > | \leq 1$

For the general case $x, y \neq 0$, set $\tilde{x} = \frac{x}{\|x\|}$, and $\tilde{y} = \frac{y}{\|y\|}$ so that $\|\tilde{x}\| = \|\tilde{y}\| = 1$ By the special case we have, $1 \geq |\langle \tilde{x}, \tilde{y}\rangle| = |\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\rangle| = \frac{|\langle x,y\rangle|}{\|x\| \cdot \|y\|}$

The important remark here is that this proof works not just for $\mathbb{R}^n$ but for any scalar product, a real-valued symmetric function of two variables that is linear in the first variable (that is, $< cx, y >= c < x, y >$ and $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$, for all x, y, z in X) and satisfies $\langle x, x\rangle > 0$ for all $x \neq 0$ in X. As another example, consider $(C[0,1])$, the space of continuous function $f : [0,1] \to \mathbb{R}$
The inner product is $\langle f, g\rangle = \int_0^1 f(x)g(x)\delta x$

Next Step: Deducing the triangle inequality from Cauchy-Schwarz:
Given $x, y \in \mathbb{R}^n$, we will prove that $\|x+y\| \leq \|x\| + \|y\|$, and this implies $\|x-z\| \leq \|x-y\| + \|y-z\|$, the triangle inequality to be proved.

$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle \leq \langle x, x\rangle + 2\|x\| \cdot \|y\| + \langle y, y\rangle$, since by Cauchy-Schwarz, $\langle x, y\rangle \leq \|x\| \cdot \|y\|$.

There are other metrics on $\mathbb{R}^n$ to be seen later. For example, $\forall p \geq 1$ $\|x-y\|_p$ and $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ are a metric.

Next to $p = 2$, the most useful metric arise from $p = 1$, and $p = \infty$, which we'll define next.

Definition 5.17. *For $x \in \mathbb{R}^n$, define $\|x\|_\infty = max|x_i|_{1 \leq i \leq n}$.*

Ball in a metric space: Define for $x \in X$ and $r > 0$, the ball $B(x, r)$ in (X, d) by $B(x, r) = \{y \in X : d(x, y) \leq r\}$.

Bounded Set: A set in a metric space is said to be bounded if it is contained in a ball.

Definition 5.18 (Open set)**.** *A Set $V \subset X$ is said to be* **open** *in X if $\forall x \in V$, $\exists r > 0$ such that $B(x, r) \subset V$.*

Example 5.19 (Examples of open sets)**.** *Examples of Open Sets: open Intervals $(a, b) \subset \mathbb{R}$, and more generally open balls $B(x, r) = \{y \in X : d(x, y) > r\}$ This follows from the triangle inequality.*

Proposition 5.20. *The intersection of two (or finitely many) open sets is open.*

Proposition 5.21. *The union of any collection of open sets is open.*

Note that the intersection of infinitely many open sets need not be open, for example $\bigcap_{n=1}^{\infty} \left(0, 1+\frac{1}{n}\right) = (0, 1]$.

Proof of Proposition 5.20. Suppose V, W are open, let $x \in V \cap W$; then $\exists r$, and an $\epsilon > 0$ with $B(x, r) \subset V$ and $B(x, \epsilon) \subset W$. Take $\gamma = \min(r, \epsilon)$, then $B(x, \gamma) \subset V \cup W$. □

Proof of Proposition 5.21. Suppose V_α for $\alpha \in J$ are open sets in X, then, V which is equal to $\bigcup_{\alpha \in J} V_\alpha$ is open. If $x \in V$, then $\exists \alpha \in J$ with $x \in V_\alpha$. Hence, $\exists v > 0$ with $B(x, v) \subset V_\alpha \subset V$. This completes the proof. □

Given a set E in a metric space (X, d), the **closure** $\tilde{E}$ of E is defined thus:
$\tilde{E} = \{x \in X : \forall r > 0 B(x, r) \cap E \neq 0\}$. In particular, $\tilde{E} \supset E$
(Note: $\tilde{E}$ is used here as closure not as complement.)

Also, Note the following in $\mathbb{R}$: • $\widetilde{(a, b)} = [a, b]$.

- $\widetilde{(a,b]} = [a,b]$.
- $\widetilde{[a,b]} = [a,b]$.

Hence, the set E is closed if $\widetilde{E} = E$.

5.3. **Homework:**

(1) For $x \in \mathbb{R}^n$, prove that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.
(2) Show that E in a metric space is closed iff $X \setminus E$ is open.
(3) For $x \in X$ and $E \subset X$, we define $d(x,E) = \inf\{d(x,y) : y \in E\}$. Show that $E = \{x \in X : d(x,y) = 0\}$.
(4) For $p > 1$ and $a, b > 0$, show that $\frac{a^p}{p} + \frac{b^q}{q} \geq ab$, where $\frac{1}{p} + \frac{1}{q} = 1$.
(5) Determine if the following sets are open or closed or neither:
   - $\mathbb{Z} \subset \mathbb{R}$ with the standard metric.
   - $\{(x,y) \subset \mathbb{R}^2 : xy \geq 1\}$
(6) Prove that $V \subset \mathbb{R}^n$ is open iff it is open for the metric $\|x - y\|_1$.

Math H104: Honors Introduction to Analysis	**Fall 2005**

Lecture 7: September 20

Lecturer: Yuval Peres *Scribe: David Wong*

We will show that the following formula for the closure of E is equivalent to the definition previously stated in the lecture notes.

$$\begin{aligned} \bar{E} &= \text{closure of } E = \{\text{all limit points of } E\} \\ \text{and } \bar{E} &= \{\lim_{n\to\infty} x_n | \{x_n\}_{n=1}^{\infty} \subset E\} \end{aligned}$$

Note that the sequences are not required to have distinct elements; thus, $x = \lim_{n\to\infty}(x, x, \ldots)$, i.e., the sequence for which $x_n = x$ for all n. Clearly, $E \subset \bar{E}$ under this definition.

Proof. ($\subset$) Let $x \in \bar{E}$. By definition, $B(x, \frac{1}{n})$ contains some point $x_n \in E$. Clearly, $x_n \to x$ as $n \to \infty$, so every point x in the closure $\bar{E}$ is a limit point of a sequence in E.

$\Big[$Note that convergence for a general metric space, $x_n \overset{n\to\infty}{\to} x$, means that $\forall \epsilon > 0, \exists n_0 : \forall n > n_0, d(x_n, x) < \epsilon$. In our proof, we take $n_0 = \lceil \frac{1}{\epsilon} \rceil$, the smallest integer larger than $\frac{1}{\epsilon}$.$\Big]$

($\supset$) Suppose $x = \lim_{n\to\infty} x_n$ with all $x_n \in E$. Then $\forall \epsilon > 0$, there exists $n_0 > n$, such that $d(x, x_n) < \epsilon$. Thus there is a ball $B(x, \epsilon)$ that intersects E; therefore, x is in the closure of E. ☐

Suppose (X, d) and (Y, ρ) are metric spaces. A function $f : X \to Y$ is **continuous at the point** $x \in X$ if $y = f(x)$ satisfies

(1) $\forall \epsilon > 0,\ \exists \delta > 0 : f(B(x, \delta)) \subset B(y, \epsilon)$.

(2) For any sequence $\{x_n\}_1^{\infty}$ in X converging to x, we have $f(x_n) \to f(x)$.

Proof. We prove that the two conditions above are equivalent.

((1) $\Rightarrow$ (2)) Given (1) is true, $x_n \to x$, and $y = f(x)$, show that $f(x_n) \to y$.
For every $\epsilon > 0$, there exists a $\delta > 0$ such that the function f maps $B(x, \delta)$ to a subset of $B(y, \epsilon)$, where $y = f(x)$. We also know that for a convergent sequence $\{x_n\}_{n=1}^{\infty}$ there is some n_0 such that for all $n > n_0$ we have $x_n \in B(x, \delta)$. This implies that $f(x_n)$, which is in $f(B(x, \delta))$ is also in $B(y, \epsilon)$. This is true for all $\epsilon > 0$, so $f(x_n) \to y$.

((2) $\Rightarrow$ (1)) Given $\epsilon > 0$ we want to find $\delta > 0$ to satisfy (1).
Let's try $\delta = 1$, otherwise $\delta = \frac{1}{2}, \ldots, \delta = \frac{1}{n}$. If for one of them $f(B(x, \frac{1}{n})) \subset B(y, \epsilon)$, we are done. We want to show these attempts could not all fail. Let's assume that they did. Then $\exists x_n \in B(x, \frac{1}{n})$, with $f(x_n) \notin B(y, \epsilon)$. But by (2) we have that $x_n \to x$. $f(x_n) \notin B(y, \epsilon)$ means $f(x_n)$ does not converge to y. Thus, our assumption is wrong, so the attempt must succeed for some δ.

□

A function $f : X \to Y$ is **continuous** if it is continuous at all $x \in X$.

Example 5.22 (Example of a piecewise continuous function that is not continuous)**.** $f(x) = \left\{ \begin{array}{cc} 1 & x > 0 \\ 0 & x \leq 0 \end{array} \right\}$. *Then $f : \mathbb{R} \to \mathbb{R}$ continuous at all x not equal to* 0.

Theorem 5.23. *$f : X \to Y$ is continuous if and only if for any open $V \subset Y$ we have $f^{-1}(V)$ open in X. (Where $f^{-1}(V) = \{x \in X : f(x) \in V\}$, which is different from textbook, and f not necessarily invertible or one to one.)*

Proof. ($\Rightarrow$) To show $f^{-1}(V)$ open, for each $x \in f^{-1}(V)$, we must find a ball centered at x and contained in $f^{-1}(V)$.

Let $V \subset Y$ be open. Let $x \in f^{-1}(V)$. $x \in f^{-1}(V)$, so $y = f(x) \in V$, whence $\exists \epsilon > 0$ with $B(y, \epsilon) \subset V$. Hence by continuity of f $\exists \delta > 0$ such that $f(B(x, \delta)) \subset B(y, \epsilon) \Rightarrow B(x, \delta) \subset f^{-1}(V)$.

($\Leftarrow$) Let $f^{-1}(V)$ be open in X for every open set $V \subset Y$. Since V is open in Y, then there exists $\epsilon > 0$ such that for any $f(x) \in Y$, $B(f(x), \epsilon) \subset Y$. Then, note that $f^{-1}(B(f(x), \epsilon))$ is open in X and it contains x. Thus, there exists a ball $B(x, \delta)$ which is a subset of $f^{-1}(B(f(x), \epsilon))$. Then, $f(B(x, \delta)) \subset B(f(x), \epsilon)$. Therefore, f is continuous in X. □

Definition 5.24. $\{V_\alpha\}_{\alpha \in J}$ **covers** A *means* $\bigcup_{\alpha \in J} V_\alpha \supset A$

Definition 5.25. *Let (X, d) be a metric space. The set $A \subset X$ is called* **compact** *if for any collection of open sets $\{V_\alpha\}_{\alpha \in J}$ that* **covers** *A there is a finite subcover $\{V_{\alpha_i}\}_{i=1}^n$.*

Definition 5.26. *$A \subset X$ is* **sequentially compact** *if for any sequence $\{x_n\}$ in A there is a convergent subsequence $x_{n_k} \overset{k\to\infty}{\to} x \in A$ (to a limit in A).*

Theorem 5.27. KEY

(1) *A compact $\iff$ A sequentially compact*
(2) *For $A \subset \mathbb{R}^m$, A compact $\iff$ A closed and bounded*

As a warm up we prove the **Heine-Borel** theorem.

Theorem 5.28. *A closed interval $I_0 = [a, b] \subset \mathbb{R}$ (where $a \leq b$) is compact.*

Proof. Suppose we are given a cover $\{V_\alpha\}_{\alpha \in J}$ of I_0 by open sets.

$$I_0 = [a, b] = \underbrace{\left[a, \frac{a+b}{2}\right]}_{I_1} \cup \underbrace{\left[\frac{a+b}{2}, b\right]}_{\tilde{I}_1}$$

If both of these have finite subcovers, we are done. Otherwise, one of them, say I_1, does not have a finite subcover. Write $I_1 = I_2 \cup \tilde{I}_2$. By assumption, one of them does not have a finite subcover, say I_2. Continuing, we get a sequence $I_n = [a_n, b_n]$ of intervals without a finite subcover $b_n - a_n = \frac{b-a}{2^n}$. (Intervals converge to a point.) Let $z = \sup a_n$ and note $z \in I_n, \forall n$. I_0 is given a cover $\{V_\alpha\}_{\alpha \in J}$, so $\exists \alpha$ with $z \in V_\alpha$. V_α open, so $\exists \epsilon$ such that

$B(z, \epsilon) \subset V_\alpha$. But there exists some n such that $b_n - a_n < \epsilon$ (namely when $\frac{b-a}{2^n} < \epsilon$). At which point, $I_n \subset B(z, \epsilon) \subset V_\alpha$. Thus, this contradicts our original assumption, therefore interval is compact.

□

Lecture 8: September 22

Lecturer: Yuval Peres *Scribe: Jonathan Goldman*

Theorem 5.29. *If $K \subset X$ is compact, then it is closed and bounded. In fact, K is totally bounded.*

Definition 5.30. *Say that K is* ***totally bounded*** *if $\forall \epsilon > 0$ it is the case that K can be covered by finitely many ϵ-balls, i.e. $\forall \epsilon > 0$ there exists n_ϵ for which there exist $x_1^{(\epsilon)}, \ldots, x_{n_\epsilon}^{(\epsilon)}$ in K such that $K \subset \bigcup_{i=1}^{n_\epsilon} B(x_i^{(\epsilon)}, \epsilon)$.*

A totally bounded set is certainly bounded. Take $\epsilon = 1$ and

$$R = \max_{i \in \{1,2,\ldots,n_1\}} d(x_1^{(1)}, x_i^{(1)}) + 1.$$

Observe that $K \subset B(x_1^{(1)}, R)$ by the triangle inequality.

Bounded vs. Totally Bounded Recall the discrete metric on $\mathbb{N}$: $d(x,y) = 1$ if $x \neq y$ and $d(x,x) = 0$. $\mathbb{N}$ under the discrete metric is bounded but not totally bounded. $B(x, 1.1)$, for example, contains all of $\mathbb{N}$, but for any $\epsilon < 1$, we would need infinitely many ϵ-balls to cover $\mathbb{N}$, namely one for each $n \in \mathbb{N}$.

A more important example of this that we'll see later: Let $X = C[0,1] = \{f : [0,1] \to \mathbb{R}, f \text{ continuous}\}$. The metric for X is $d(f,g) = \max |f(x) - g(x)|, x \in [0,1]$. Then let $K = \overline{B(0,1)} = \{f : \max |f| \leq 1\}$. Then K is bounded but not totally bounded for the following reason (which holds in any metric space).

If a set K in a metric space has infinitely many points $z_1, z_2, \ldots$ with $d(z_i, z_j) \geq r > 0$, then K is not totally bounded: every ball of radius $\frac{r}{2}$ can cover at most one z_i.

Proof of 5.29. We are given K compact.

(1) K is totally bounded: Let $\epsilon > 0$. Certainly K is covered by $\bigcup_{x \in K} B(x, \epsilon)$. Then just take a finite subcover, which we are guaranteed to have by compactness. Thus $K \subset \bigcup_{i=1}^{n} B(x_i, \epsilon)$.

(2) K is closed: Suppose $x \in X$ with $d(x, K) = 0$. (Recall that this means that $d(x,K) = \inf\{d(x,y) : y \in K\}$.) That is, every $\epsilon > 0$ is such that $B(x, \epsilon) \cap K \neq \emptyset$. We need to show that $x \in K$. Suppose that it is not. Then for every $y \in K$, we have $r_y = d(x,y) > 0$ and $K \subset \bigcup_{y \in K} B(y, \frac{r_y}{2})$. The compactness of K therefore implies that there exists n such that $K \subset \bigcup_{i=1}^{n} B(y_i, \frac{r_{y_i}}{2})$. Pick $\epsilon = \frac{1}{4} \min_{1 \leq i \leq n} r_{y_i}$. Then there exists

$z \in B(x, \epsilon) \cap K$, which implies that there exists $z \in B(y_i, \frac{r_{y_i}}{2})$. We now have

$$d(x, y_i) \leq d(x, z) + d(z, y_i) \leq \frac{r_{y_i}}{4} + \frac{r_{y_i}}{2} < r_{y_i}$$

This gives the desired contradiction, and thus K is closed.

□

Recall that K is **sequentially compact** if whenever $\{x_n\} \subset K$, there is a convergent subsequence $x_{n_j} \to L \in K$.

This implies that K is closed: Assume $K \ni x_n \to x \in X$. We need to show that $x \in K$. This is easy: we know $\exists \{n_j\} : x_{n_j} \to L \in K$. Thus $d(L, x) \leq d(L, x_{n_j}) + d(x_{n_j}, x)$. But both $d(L, x_{n_j})$ and $d(x_{n_j}, x)$ go to 0 as $j \to \infty$. Thus the inequality tells us $d(L, x) \leq 0$ which implies that $d(L, x) = 0$. Since d is a metric, this means we must have $x = L$, and thus $x \in K$.

Theorem 5.31 (Key Theorem:)**.** *Suppose $f : X \to Y$ is continuous and onto, and X is compact. Then Y is compact.*

Proof. We are given an open cover of Y: $Y \subset \bigcup_{\alpha \in J} V_\alpha$. We know that $f^{-1}(V_\alpha)$ is open for every $\alpha \in J$ (by continuity). Also, $X \subset \bigcup_{\alpha \in J} W_\alpha$, where $W_\alpha = f^{-1}(V_\alpha)$. Since X is compact, $\exists \alpha_1, \ldots, \alpha_N$ with $X \subset \bigcup_{i=1}^{N} W_{\alpha_i}$. Since f is onto, $\forall y \in Y$ there exists $x \in X$ such that $f(x) = y$. Now there exists i with $x \in W_{\alpha_i}$, which implies that $f(x) = y \in V_{\alpha_i}$. Therefore $Y \subset \bigcup_{i=1}^{N} V_{\alpha_i}$. □

Corollary 5.32. *Suppose $f : X \to Y$ is continuous and $K \subset X$ is compact. Then $f(K)$ is compact, and in particular, closed and bounded.*

Corollary 5.33. *Suppose $f : X \to \mathbb{R}$ is continuous and $K \subset X$ is compact. Then f is bounded on K and f attains its maximum and minimum on K.*

What is meant by f attaining its maximum and minimum on K? Let $M = \sup_K f = \sup\{f(x) : x \in K\} < \infty$. If there exists $x_n \in K$ such that $M - \frac{1}{n} < f(x_n) \leq M$, then $M \in \overline{\{f(x) : x \in X\}} = \overline{f(K)}$. This implies that $M \in f(K)$, which is what is meant by f attaining its max on K, i.e. $\exists x_* \in K : f(x_*) = M = \max_{x \in K} f(x)$.

5.4. **Homework.**

(1) Show $f_n(x) = \sin(n\pi x)$ for $n = 1, 2, 3, \ldots$ satisfy $d(f_n, f_m) \geq r > 0$ for all $n \neq m$. Hint: Show that $\int_0^1 |f_n(x) - f_m(x)|^2 = a > 0$ for $n \neq m$.

(2) Prove directly from the definitions that if X is sequentially compact and $f : X \to Y$ is continuous and onto, then Y is sequentially compact. (Recall the sequence definition of continuity for this problem.)

(3) Problem 40 from the book, page 119.

(4) Give an example of a continuous $f : \mathbb{R} \to \mathbb{R}$ and open subset $V \subset \mathbb{R}$ with $f(V)$ not open, or prove that no such V exists.

Math H104: Honors Introduction to Analysis	**Fall 2005**
Lecture 9: September 27	
Lecturer: Yuval Peres	*Scribe: Ke Lu*

Holder's inequality in $\mathbb{R}^n$: For x, y in $\mathbb{R}^n$ write $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. If $\frac{1}{p} + \frac{1}{q} = 1$ where $1 \leq p < \infty$ and $1 \leq q < \infty$, then

$$\sum_{i=1}^n |x_i y_i| \leq \|x\|_p \|y\|_q \tag{3}$$

Check for $p = 1$ and $q = \infty$.

$$\begin{aligned} \sum_{i=1}^n |x_i y_i| &\leq \sum_{i=1}^n |x_i| max_j |y_j| \\ &\leq \|x\|_1 \|y\|_\infty \end{aligned}$$

For the case of $p = 2$; it boils down to Cauchy-Schwarz.

Proof. First assume $\|x\|_p = \|y\|_q = 1$. Then by an earlier exercise

$$|x_i y_i| \leq \frac{|x_i|^p}{p} + \frac{|y_i|^q}{q}.$$

Summing over i, we get

$$\begin{aligned} \sum |x_i y_i| &\leq \frac{||x||_p^p}{p} + \frac{||y||_q^q}{q} \\ &\leq \frac{1}{p} + \frac{1}{q} \\ &= 1. \end{aligned}$$

For general x and y (both assumed nonzero, otherwise trivial), let $\tilde{x} = \frac{x}{||x||_p}$ and $\tilde{y} = \frac{y}{||y||_p}$. Now $||\tilde{x}||_p = 1$ and $||\tilde{y}||_q = 1$. By the special case proved, we get $\sum |\tilde{x}_i \tilde{y}_i| \leq 1$. Hence $\sum \frac{|x_i y_i|}{\|x\|_p \ \|y\|_q} \leq 1$. Therefore, $\sum |x_i y_i| \leq \|x\|_p \|y\|_q$. ◻

Definition 5.34. *V is a vector space if it has a commutative and associative addition operation and for v, w in V and c in $\mathbb{R}$. $c \cdot v$ is defined and $c(v + w) = cv + cw$.*

Examples of vector spaces are $\mathbb{R}^n$, the set $C[0, 1]$ of continuous functions from $[0, 1]$ to R, and $B[0, 1]$ the set of bounded functions from $[0, 1]$ to R.

A norm $\| \cdot \|$ is a function from a vector space to $[0, \infty]$ such that

(1) $\|v\| = 0$ if and only if $v = 0$.
(2) $\|cv\| = |c| \cdot \|v\|$ for every c in $\mathbb{R}$ and v in V.
(3) $\|v + w\| \leq \|v\| + \|w\|$ for every v, w in V.

Given a norm $\|\cdot\|$ on V, we get a metric on V: $d(v,w) = \|v-w\|$.

In $\mathbb{R}^n$, $\|x\|_2$ is the Euclidean norm. $\|x\|_\infty = \max_{1\le i\le n}|x_i|$ is another norm.

For $1 \le p < \infty$, the triangle inequality for $||\ ||_p$ is called Minkowski's inequality.

Proposition 5.35 (Minkowski's inequality)**.** $\forall x,\ y \in \mathbb{R}^n,\ ||x+y||_p \le ||x||_p + ||y||_p$

Proof.

$$\begin{aligned}
\|x+y\|_p^p &= \sum_{i=1}^n |x_i+y_i|^p \\
&= \sum_{i=1}^n |x_i+y_i| \cdot |x_i+y_i|^{p-1} \\
&\le \sum |x_i| \cdot |x_i+y_i|^{p-1} + \sum |y_i| \cdot |x_i+y_i|^{p-1} \\
&\le \|x\|_p \cdot \|\{|x_i+y_i|^{p-1}\}_{i=1}^n\|_q + \|y\|_p \cdot \|\{|x_i+y_i|^{p-1}\}_{i=1}^n\|_q.
\end{aligned}$$

Now

$$\begin{aligned}
\|\{|x_i+y_i|^{p-1}\}_{i=1}^n\|_q &= (\sum |x_i+y_i|^{(p-1)q})^{1/q} \\
&= (\sum |x_i+y_i|^p)^{1/q} \\
&= \|x+y\|_p^{p/q}.
\end{aligned}$$

We assume $x+y \neq 0$. Divide both sides by $\|x+y\|_p^{p/q}$, get $\|x+y\|_p \le \|x\|_p + \|y\|_p$. ☐

Now we return to topology.

Definition 5.36. *A metric space X is* **complete** *if every Cauchy sequence in X converges to a limit in X.*

We have shown that $\mathbb{R}$ is complete. On the other hand, $\mathbb{R}\setminus\mathbb{Q}$ and $\mathbb{Q}$ are not. Here is a general fact.

Proposition 5.37. *If $(X,\ d)$ is complete and Y is a subset of X then Y is complete if and only if Y is closed in X.*

Proof. Given Y is closed. Take any Cauchy sequence $\{y_n\}$ of Y. $\{y_n\}$ converges to x in X. x must be in Y then.

Given Y is complete. Take any Cauchy sequence $\{y_n\}$ of Y that converges to x in X, we can construct a subsequence that is Cauchy. Thus, x in Y. ☐

$\mathbb{R}^k$ with the usual metric is complete. Take a Cauchy sequence $\{x^{(n)}\}_n$ of $\mathbb{R}$. If $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, ...x_k^{(n)})$, then $\{x_i^{(n)}\}_n$ is Cauchy in $\mathbb{R}$ becase $|x_i^{(n)} - x_i^{(m)}| \le \|x^{(n)} - x^{(m)}\|_2$. By completeness of $\mathbb{R}$, we get $\lim_{n\to\infty} x_i^{(n)} \to x_i^*$. Let $x^* = (x_1^*,\ x_2^*,\ x_3^*,\ \ldots x_k^*)$. Claim: $\lim_{n\to\infty} x^{(n)} \to x^*$.

$$\lim_{n\to\infty} \|x^{(n)} - x^{(m)}\|_2 = \sqrt{\sum_{i=1}^k (x_i^{(n)} - x_i^*)^2} = 0$$

Example 5.38. *The metric space* $C[0,1]$ *with distance function* $\|f\|_\infty = \max_{0\le x\le 1}|f(x)|$ *is complete.*

Fact 5.39. *A limit of uniformly converging sequence of continuous functions is continuous. (Verify)*

We say f_n uniformly converges to f if $\|f_n - f\|_\infty = 0$.

Given an infinite subset A of X, the metric space we say that the point x is an accumulation point of A if for all $r > 0$ the intersection of $B(x,r)$ and A is infinite.

Examples: $\mathbb{Z} \subset \mathbb{R}$ has no accumulation points. For $\mathbb{Q} \subset \mathbb{R}$ every element is an accumulation point. For the sequence $\{1/n\}$, 0 is the only accumulation point.

Theorem 5.40. *X is sequentially compact if and only if every infinite subset A of X has an accumulation point in X.*

Proof. Assume X is sequentially compact. A contains some sequence $\{x_n\}$ with all x_n distinct. There exists a subsequence $\{x_{n_k}\}$ converging to x in X. x is an accumulation point.

Assume every infinite subset has an accumulation point. Take any sequence $\{x_n\}$ in X. If some element z is repeated infinitely many times, then we can construct a subsequence that only contains z, which clearly converges to z. Otherwise, we can find a subsequence $\{x_{n_k}\}$ of distinct elements. Let A be the set containing all of the elements $\{x_{n_k}\}$. A is infinite and thus has an accumulation point L in X. This means that for all $j \in \mathbb{N}$, there exists n_{k_j} such that $x_{n_{k_j}}$ is in $B(L, 1/j)$. $\{x_{n_{k_j}}\}$ converges to L. □

Theorem 5.41. *If $(X,\ d)$ is a metric space then the following are equivalent.*

(1) *X is compact.*
(2) *X is sequentially compact.*
(3) *X is complete and totally bounded.*

We will only show **1.** implies **2.** in this lecture.

Proof. Given X compact and an infinite subset S of X, we need to show there is an accumulation point for S. If there is no accumulation point then for every x in X, there exists $r_x > 0$ where $B(x, r_x) \bigcap S$ is finite. Trivially, for all x in X, x is in $B(x, r_x)$. It follows that $\{B(x, r_x)\}$ form an infinite open cover of X; Then there must a finite subcover $\bigcup_{i=1}^N B(x_i, r_{x_i})$. Then $\bigcup_{i=1}^N (B(x_i, r_{x_i}) \bigcap S)$ must equal S. S is then an union of a finite number of finite sets so S must also be finite, which is a contradiction. □

Lecture 10: September 29

Lecturer: Yuval Peres *Scribe: Lucas Parker*

Proposition 5.42. *For a metric space (X, d) the following are equivalent.*

(1) *X is compact.*
(2) *X is sequentially compact.*
(3) *X is totally bounded and complete.*

We have already established that (1)$\Leftrightarrow$ (2).

Proposition 5.43. *(2)$\Rightarrow$ (3)*

Proof. Let X be a sequentially compact metric space. Set $\epsilon > 0$. Choose $\{x_j\}_{j\geq 1}$ as follows: $x_1 \in X$ is arbitrary, and x_2 is such that $d(x_2, x_1) \geq \epsilon$, if possible. Otherwise, set $N = 1$. Continue by induction. If $x_1, x_2, \ldots, x_k$ are already chosen, find x_{k+1} such that $d(x_{k+1}, x_i) \geq \epsilon\ \forall 1 \leq i \leq k$ if possible. Otherwise, set $N = k$ and stop. If we can pick x_{k+1} for each k, we get an infinite sequence $\{x_k\}_{k=1}^{\infty}$ and a contradiction: The set $\{x_k\}_{k=1}^{\infty}$ has no accumulation point because $\forall z \in X$, $B(z, \epsilon/2)$ can only contain at most one x_k. So the procedure must have stopped, and so N is finite, and $\cup_{i=1}^{N} B(x_i, \epsilon) = X$.

Now, to prove that X is complete: Given any Cauchy sequence $\{x_j\}_{j=1}^{\infty}$, there must exist some subsequence $\{x_{j_k}\}_{k=1}^{\infty}$ that converges to a limit $x_* \in X$. So, $\forall \epsilon > 0$, there exists some k_0 such that $\forall k > k_o$, we have $d(x_{j_k}, x_*) < \epsilon/2$. Also, $\exists N$ such that $\forall m, n \geq N$, $d(x_m, x_n) < \epsilon/2$. Let $J = \max(j_{k_o}, N)$. Fix some $\tilde{J} = j_{k_1} > J$. Then for any $j \geq J$, we obtain $d(x_j, x_*) \leq d(x_j, x_{\tilde{J}}) + d(x_{\tilde{J}}, x_*)$, which is less than ϵ. ☐

Proposition 5.44. *3) $\Rightarrow$ 1)*

Proof. Let X be totally bounded and complete, and let $\{V_\alpha\}_{\alpha\in S}$ be an open cover of X. We will proceed with a proof by contradiction:

Suppose there is no finite subcover. X is totally bounded, so $X = \cup_{j=1}^{N_1} B(x_j^{(1)}, 2^{-1})$, or $X = \cup_{j=1}^{N_k} B(x_j^{(k)}, 2^{-k})$ for all k. $\exists l_1 \leq N_1$ where for $B(x_{l_1}^{(1)}, 2^{-1}) = K_1$ there is no finite subcover from $\{V_\alpha\}_{\alpha\in S}$ (If every ball had a finite subcover, then there would be a finite subcover of X). K_1 is also covered by $\cup_{j=1}^{N_2} B(x_j^{(2)}, 2^{-2})$. So, there is some l_2 such that $K_2 = K_1 \cap B(x_{l_2}^{(2)}, 2^{-2})$ has no finite subcover. Continue by induction: Given K_m contained in a ball of radius 2^{-m} such that it has no finite subcover from $\{V_\alpha\}_{\alpha\in S}$, define K_{m+1} as follows: $K_m \subset \cup_{j=1}^{N_{m+1}} B(x_j^{(m+1)}, 2^{-(m+1)})$. So, there is some l_{m+1} where $K_{m+1} = K_m \cap B(x_{l_{m+1}}^{(m+1)}, 2^{-(m+1)})$ has no finite subcover.

Now, consider the sequence $\{x_{l_m}^{(m)}\}_{m=1}^{\infty}$. $x_{l_m}^{(m)} \in B(x_{l_n}^{(n)}, 2^{1-n})$ when $n < m$. $\{x_{l_m}^{(m)}\}$ is cauchy, as $\forall \epsilon > 0$, take M such that $2^{-M} < \epsilon/2$, then $\forall\ n, m \geq M$, $d(x_{l_m}^{(m)}, x_{l_n}^{(n)}) < \epsilon$. As X is complete, $x_{l_m} \longrightarrow x \in X$ as $m \longrightarrow \infty$. As $x \in X$, there is some α such that $x \in V_\alpha$. As V_α

is open, $\exists \epsilon > 0$ with $B(x, \epsilon) \subset V_\alpha$. Choose m such that $2^{-m} < \epsilon/2$. So $K_m \subset B(x_{l_m}^{(m)}, 2^{-m})$, and so $K_m \subset B(x, \epsilon) \subset V_\alpha$, which is clearly a contradiction to our initial assumption. $\square$

Proposition 5.45. *If $K \subset \mathbb{R}^n$, then K is closed and bounded $\Rightarrow$ K is compact.*

6. Homework

(1) Prove Proposition 5.45) using all 3 notions of compactness in homework groups. Only one needs to be turned in.
(2) Suppose (X, d) is totally bounded, and $Y \subset X$, show (Y, d) is totally bounded.
(3) Given (X, d), show X is complete iff for every nested sequence of closed sets $\{K_m\}_{m=1}^{\infty}$, $K_m \neq \emptyset$, and such that diameter$(K_m) \to 0$, we have that $\cap_{m=1}^{\infty} K_m \neq \emptyset$.
(4) Show X is compact iff every nested sequence of closed, non-empty sets in X has a non-empty intersection.

Math H104: Honors Introduction to Analysis **Fall 2005**

Lecture 11: October 04

Lecturer: Yuval Peres *Scribe: Vuko Buckovic*

Definition: The topology of a metric space (X, d) is the collection of open sets in X. More generally, a topology in the set X is any collection W of subsets of X, called "the open sets", which satisfy the following:

(1) $\emptyset \in W, X \in W$.
(2) $V_\alpha \in W$, for $\alpha \in J \Rightarrow \bigcup_{\alpha \in J} V_\alpha \in W$.
(3) $V_1, V_2 \in W \Rightarrow V_1 \cap V_2 \in W$.

Often one adds more requirements, for instance the topological space (X, W) is called a **Hausdorff space** if for each $x, y \in X$ there are open and disjoint sets U, V such that $x \in U$ and $y \in V$ (Any two points can be separated by two disjoint sets). Clearly, the notions of closed set, compact set and accumulation point depend on the topology. For example, consider the function $f : X \to Y$ for which the inverse $f^{-1}(V)$ is open for all open $V \subset Y$. Also, $x_n \to x$ in a topological space means that for each open set V with $x \in V$ (V neighbourhood of x) there exists N such that for every $n \geq N$ $X_n \in V$.

So, what properties are not affected when we change the metric?

Example 6.1. *Suppose that $X = \mathbb{R}^k$ with the metric $d_p(x, y) = \|x - y\|_p$ where $1 \leq p \leq \infty$. These metrics yield the same topology on $\mathbb{R}^k$.*

Consider a set X with two metrics, d_1 and d_2 which yield topologies W_1 and W_2. We can write $B_1(x, r)$ for balls in d_1 and $B_2(x, r)$ for balls in d_2. Then $W_2 \subset W_1$ is equivalent to $\{\forall x \in X$ and $\forall r > 0$ $\exists \varepsilon > 0$ such that $B_1(x, \varepsilon) \subset B_2(x, r)\}$ which is immediate from definitions (it is left as the exercise to check it!). Both statements are equivalent to saying that the identity mapping $(X, d_1) \to (X, d_2)$ is continuous.

We can now return to the case $X = \mathbb{R}^k$. We can choose reference metric (it is enough to check that $\| \|_p$ and $\| \|_\infty$ yield the same metric: $\|x\|_p = (\sum_{i=1}^k \|x_i\|^p)^{\frac{1}{p}} \leqslant k^{\frac{1}{p}} \|x\|_\infty$). In the case of the balls that means that $B_p(z, r) \subset B_\infty(z, r) \subset B_p(z, k^{\frac{1}{p}} r)$. Larger metric means, generally, that the ball is smaller; $\| \|_p$ and $\| \|_\infty$ have a stronger relation than just to yield the same topology; they have a direct relation.

Question: If d_1 and d_2 are metrics on X that yield the same topology and $\{x_n\} \subset X$ is a Cauchy sequence in d_1 is it the sequence Cauchy in d_2? The answer is no and it is explained in the following example.

Example: Let $X = (0, 1)$ and define $d_1(x, y) = |x - y|$ and $d_2(x, y) = |\frac{1}{x} - \frac{1}{y}|$. With the metric d_1 in (0, 1) $x_n \to x$ is equivalent to $\frac{1}{x_n} \to \frac{1}{x}$ in $(1, \infty)$ with the usual metric while $x_n \to x$ in $(0, 1)$ with d_2.

Definition: Let (X, d) and (Y, ρ) be metric spaces. The function $f : X \to Y$ is uniformly continuous in X if for each $\varepsilon > 0$ $\exists \delta > 0$ such that $d(x_1, x) < \delta$, implies that $\rho(f(x_1), f(x)) < \varepsilon$. Alternatively, expressed in usual terms, $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $\forall x \in X$, we have $f(B(x, \delta)) \subset B(f(x), \varepsilon)$. Very similar to the definition of the continuity! For instance, $f : (0, 1) \to (1, \infty)$ defined by $f(x) = \frac{1}{x}$ is continuous in $(0, 1)$ but not uniformly. Take $\varepsilon = \frac{1}{2}$, for any candidate $\delta > 0$ we can find $\alpha|\frac{1}{n} - \frac{1}{m}| < \delta$ but $|f(\frac{1}{n}) - f(\frac{1}{m})| \geqslant 1$.

Theorem 6.2. *(General Topology) Suppose X is a Hausdorff topological space. Then every compact subset $K \subset X$ is closed (in X).*

Proof. Fix $z \in X \setminus K$. For each $x \in K$ there exist disjoint sets U_x and V_x, both open in X, such that $x \in U_x$ and $z \in V_x$. The collection $\{U_x \cap K\}_{x \in K}$ is an open cover of K. Since K is compact, it has a finite subcover $\{U_{x_i} \cap K\}_{i=1}^n$. The finite intersection $V^*(z) = \cap_{i=1}^n V_{x_i}(z)$ is an open set containing z. Clearly $V^*(z)$ is disjoint from $U_{x_i}(z)$ for all $1 \leqslant i \leqslant n$, so $V^*(z)$ is disjoint from K. Finally, the union $\cup_{z \in X \setminus K} V^*(z) = X \setminus K$ is open in X. □

Theorem 6.3. *Suppose (X, d) and (Y, p) are metric spaces and $f : X \to Y$ is a continuous function on X. If X is compact, then the function f is uniformly continuous on X.*

Proof. Given $\varepsilon > 0$ we know that for each $x \in X$ there is $\delta_x > 0$ such that $f(B_d(x, \delta_x)) \subset B_\rho(f(x), \varepsilon)$ (we used continuity point by point where δ_x depends on x). Balls $\{B_d(x, \frac{\delta_{x_i}}{2})\}_{x \in X}$ are open cover of X and by compactness there is a finite subcover $\{B_d(x_i, \frac{\delta_{x_i}}{2})\}_{i=1}^n$ (because all δ_{x_i} will not work for all x_i 's as centers). So, we have that $\forall x \in X$ $\exists i$ such that $x \in B_d(x_i, \frac{\delta_{x_i}}{2})$. Set $\delta_* = \min\{(\frac{\delta_{x_i}}{2}) : 1 \leq i \leq n\}$. Then and $B_d(x, \delta_*) \subset B_d(x_i, \frac{\delta_{x_i}}{2} + \delta) \subset B_d(x_i, \delta_i)$. Therefore, $f(B_d(x, \delta_*) \subset B_\rho(f(x_i), \varepsilon)$ from where it follows that $\rho(f(x), f(x_i)) < \varepsilon$ and from these two facts we can conclude that $f(B_d(x, \delta_*) \subset B_\rho(f(x), 2\varepsilon)$. □

Math H104: Honors Introduction to Analysis **Fall 2005**

Lecture 12: October 06

Lecturer: Yuval Peres *Scribe: Cordelia Csar*

Definition 6.4. *Given metric spaces X and Y, the function $f : X \to Y$ is a* homeomorphism *if f is continuous, onto and has a continuous inverse $g : Y \to X$ with $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$.*

Such a map preserves all "topological" properties. A property of a topological space is a topological property if it is preserved under homeomorphism.

There is a bijection between $[0, 1]$ and $[0, 1]^2$. Why? There exists an injection $[0, 1] \to [0, 1]^2$ and an injection $[0, 1]^2 \to [0, 1]$. The details of this second map are as follows. Write $x = \sum_{k=0}^{\infty} x^k 10^{-k}$ with $0 \le x_k \le 9$ and the decimal expansion not terminating in an infinite sequence of 9s. If $x = 1$ then $x_0 = 1$ and $x_k = 0$ for $k > 0$. $y = \sum_{k=0}^{\infty} y^k 10^{-k}$ with the same properties as x above.

Define the function $h : [0, 1]^2 \to [0, 1]$ by

$$h(x, y) = 0.x_0y_0x_1y_1 \ldots = \sum_{k=0}^{\infty} x_k 10^{-(2k+1)} + \sum_{k=0}^{\infty} y_k 10^{-(2k+2)}.$$

$h : [0, 1]^2 \to [0, 1]$ is not onto, but it can be easily be verified that h is one-to-one. Then by the Schroeder-Bernstein Theorem, there exists a bijection $[0, 1] \to [0, 1]^2$. The Peano Curve is a function $f : [0, 1] \to [0, 1]^2$ which is onto and continuous. (Note: There is no homeomorphism $[0, 1] \to [0, 1]^2$.)

Lemma 6.5. *Suppose X is compact and $f : X \to Y$ is continuous, onto and one-to-one. Then f is a homeomorphism.*

Proof. We can define $g : Y \to X$ with $g(y) = x$ if $f(x) = y$. Given an open $V \subset X$, we need to check that $g^{-1}(V)$ is open in Y. $K = X \setminus V$ is closed in X, hence it is compact. Therefore, $f(K)$ is also compact. $f(K) = g^{-1}(K)$. Then $g^{-1}(V) = g^{-1}(X \setminus K) = g^{-1}(X) \setminus g^{-1}(K)$ which is open in Y. □

Definition 6.6. *A (topological or) metric space X is* path-connected *if for any $x, y \in X$ there exists a path connecting them, i.e., $\exists \gamma : [0, 1] \to X$, which is continuous with $\gamma(0) = x$ and $\gamma(1) = y$.*

Clearly, $\mathbb{R}^k$ and $[a, b]$ are connected. More generally, if $K \subset \mathbb{R}^\ell$ is *convex*, it is path-connected. K is convex if for all $x, y \in K$ and for all $t \in [0, 1]$, $\gamma(t) = ty + (1 - t)x \in K$.

Definition 6.7. *A set $K \subset X$ is called* clopen *if it is closed and open in X.*

Definition 6.8. *X is* connected *if the only clopen sets in X are $\emptyset$ and X.*

Equivalently, X is connected if and only if for any partition $X = V_1 \cup V_2$ of X into two open and disjoint sets, one of them is $\emptyset$. For instance, $\mathbb{Q} = \{x \in \mathbb{Q} : x < \sqrt{2}\} \cup \{x \in \mathbb{Q} : x > \sqrt{2}\}$ and thus $\mathbb{Q}$ is not connected.

Fact 6.9. *For $a < b \in \mathbb{R}$, $[a, b]$ is connected.*

Proof. Suppose $K \subset [a, b]$ is clopen in $[a, b]$ and $K \neq \emptyset$, $K \neq [a, b]$. If $a \in K$, let $s = \sup K \subset [a, b]$. Then $s \in K$ because K is closed. We have three cases:

(1) Suppose $s = a$. This is impossible because we know, since K is open, that there exists $\epsilon > 0$ such that $[a, a + \epsilon) \subset K$.
(2) Suppose $s < b$. This is impossible because we know, since K is open, that there exists $\epsilon > 0$ such that $[s, s + \epsilon) \subset K$.
(3) Suppose $s = b$. This implies that $b \in K$. To get a contradiction, we examine $\sup K^c$.
    (a) Suppose $\sup K^c < b$. We apply the argument in (2) and reach a contradiction.
    (b) Suppose $\sup K^c = b$. Then $b \in K^c$. But $b \in K$, so we have a contradiction.

□

Theorem 6.10. *Suppose that X is path-connected. Then X is connected.*

Proof. Assume X is not connected. Suppose $X = V_1 \cup V_2$ with V_1, V_2 open, disjoint and nonempty. Let $x \in V_1$, $y \in V_2$. We find $\gamma : [0, 1] \to X$ with γ continuous and $\gamma(0) = x$ and $\gamma(1) = y$. Then $[0, 1] = \gamma^{-1}(V_1) \cup \gamma^{-1}(V_2)$ is a disjoint union of open sets and nonempty since $0 \in \gamma^{-1}(V_1)$ and $1 \in \gamma^{-1}(V_2)$. This is a contradiction since $[0, 1]$ is connected (meaning one of $\gamma^{-1}(V_1)$ and $\gamma^{-1}(V_2)$ has to be empty.) □

Fact 6.11. $[0, 1]$ *and the circle $C = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ are not homeomorphic.*

Proof. $[0, 1] \setminus \frac{1}{2}$ is not connected, nor path-connected, since it is a disjoint union of open sets, but $C \setminus \{z\}$ is connected for any z. T see this, let $z = (\cos\theta, \sin\theta)$. Then define $\tilde{\gamma}(t) = (\cos(\theta + t), \sin(\theta + t))$, $0 \le t \le 2\pi$. Suppose $\tilde{\gamma}(t_0) = u$ and $\tilde{\gamma}(t_1) = v$. $\tilde{\gamma} : [t_0, t_1] \to C \setminus \{z\}$. $\tilde{\gamma}$ can then be tweaked to a γ that satisfies our requirements. □

Fact 6.12. *If X is connected and $f : X \to Y$ is continuous and onto, then Y is connected.*

Proof. If K is clopen in Y then $D = f^{-1}(K)$ is clopen in X. If $D = \emptyset$ then $K = \emptyset$ while if $D = X$ then $K = Y$. □

7. Homework

(1) For each of the following spaces decide if it compact, complete and/or connected.
    (a) $\{(x, y) \in \mathbb{R}^2 : xy \ge 1\} \subset \mathbb{R}^2$.
    (b) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 1\} \subset \mathbb{R}^3$.
    (c) $\{(0, y) \in \mathbb{R}^2 : -1 \le y \le 1\} \cup \{(x, \sin\frac{1}{x}) : 0 \le x \le \frac{1}{\pi}\} \in \mathbb{R}^2$.
(2) Prove that $[0, 1]^2$ is not homeomorphic to $[0, 1]$.
(3) Prove that $[0, 1]^2$ is not homeomorphic to the circle C (above).

Math H104: Honors Introduction to Analysis **Fall 2005**

Lecture 13: October 11

Lecturer: Yuval Peres *Scribe: Bernard Liang*

Example 13.1: For $A, B \subset \mathbb{R}^n$ write $A + B = \{a + b \mid a \in A, b \in B\}$. Which of the following are true?

- A, B are compact $\Rightarrow A + B$ is compact.
- A is compact, B is closed $\Rightarrow A + B$ is closed.
- A, B are closed $\Rightarrow A + B$ is closed.

Note: $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

Proof of Part I: If x_n is a sequence in $A + B$, write, $x_n = a_n + b_n$. By compactness of A, B, we can find a common subsequence n_k such that $a_{n_k} \to a \in A$ and $b_{n_k} \to b \in B$. Then $x_{n_k} \to a + b \in A + B$. Thus $A + B$ is sequentially compact and hence compact.

Proof of Part II: Let $x_n \in A + B$ and $x_n \to x$. We want to show that $x \in A + B$. Write $x_n = a_n + b_n$. Then by compactness of A, we can find a subsequence such that $a_{n_k} \to a \in A$. Then using $a_n + b_n \to x$, we find that b_{n_k} also converges, say to b. Because B is closed, $b \in B$. Thus $(a_{n_k}, b_{n_k}) \to (a, b) \in A \times B$. Hence $x_n \to a + b$. This means that $x = a + b \in A + B$.

Theorem 7.1. *(Tychonov's theorem) If $\{X_\alpha\}_{\alpha \in J}$ are compact topological spaces, then the product $\prod_{\alpha \in J} X_\alpha$ is compact.*

Counterexample to Part III: Consider the closed subsets of $\mathbb{R}^n$: $A = \{(x, y) : xy \geq 1\}$, $B = \{(0, y) : y \in \mathbb{R}\}$. Then the set $A + B = \mathbb{R}^2 \setminus B$, which is not closed.

Alternate proof of Part I: We know that $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact. Define the map $f : X \times Y \to X + Y : f(a, b) \to a + b$. Since addition is continuous, $X + Y$ is compact.

Example 13.2: Let's try to prove Part II using open covers. We want to show that $(A + B)^c$ is open. Suppose $z \notin A + B$. Our goal is then to find an $\epsilon > 0$ such that $\forall a \in A, B(z, \epsilon) \in (A + B)^c$ is not in the closed set $a + B$. Since $a + B$ is closed, its complement is open. Thus, there exists an ϵ_a such that $B(z, \epsilon_a) \subset (a + B)^c$, and $d(z, a + B) \geq \epsilon_a$, which can be rewritten as follows: $d(z, a + B) \geq \epsilon_a \Rightarrow |z - (a + b)| \geq \epsilon_a \Rightarrow |(z - b) + a| \geq \epsilon_a \Rightarrow d(z - B, a) \geq \epsilon_a$. So $z - B$ is disjoint from $B(a, \epsilon_a)$.
We have found an open cover $\{B(a, \frac{\epsilon_a}{2})\}$, so it remains to prove existence of a finite subcover. Let $\epsilon = \min_{1 \leq i \leq N} B(a, \frac{\epsilon_a}{2})$. $\forall a \in A$, find a_i with $a \in B(a_i, \frac{\epsilon_a}{2})$ such that $|z - (a + b)| \geq \underbrace{|z - (a_i + b)|}_{\geq \epsilon_{a_i}} - \underbrace{|a - a_i|}_{\leq \frac{\epsilon_{a_i}}{2}} \geq \frac{\epsilon_{a_i}}{2} \geq \epsilon$.

Theorem 13.2: Let $V \subset \mathbb{R}^n$ be open in $\mathbb{R}^n$ and connected. Then V is path-connected.
Proof: Suppose $x \in V$. Let $W = \{y \in V : \exists \text{ path connecting } x \text{ to } y \text{ in } V\}$. $x \in W$, so

$W \neq \emptyset$. Also, W is open: $y \in W \Rightarrow \exists \epsilon > 0 : B(y, \epsilon) \subset V$, but that means that $B(y, \epsilon) \subset W$, since balls are path-connected.
$V \setminus W$ is open: If $y \in V \setminus W$, then $\exists \delta > 0$ such that $B(v, \delta) \subset V$. Then $B(v, \delta) \subset V \setminus W$ because if there is a path $x \to u \in B(v, \delta)$, then we can get a path $x \to v$. Connectedness of V implies that the only clopen sets in V are $\emptyset$ and V. But $W \subset V$ is open, so $V \setminus W$ must be closed, and since W is nonempty, $V \setminus W$ must necessarily be $\emptyset$.

Math H104: Honors Introduction to Analysis **Fall 2005**

Lecture 14: October 13

Lecturer: Yuval Peres *Scribe: Cristobal Lemus*

Definition 7.2. *Given $A \subset X, X$ a metric space. The **interior** A° of A, is the set $\{x \in A : \exists \epsilon > 0$ with $B(x, \epsilon) \subset A\}$*

If A is open in X, then $A^\circ = A$.

Fact 7.3. $A^\circ = (\overline{A^c})^c$ *check!*

Definition 7.4. *The **boundary** of A is $\partial A = \overline{A} \setminus A^\circ = \{x \in X : \forall \epsilon > 0\ B(x, \epsilon)$ intersects both A and $A^c\}$*

Example 7.5. *In $\mathbb{R}$, $\partial(a, b) = \partial(a, b] = \partial[a, b] = \{a, b\}$*

Definition 7.6. *Let V be an open set in $\mathbb{R}^d$. We call a continuous function $u : V \to \mathbb{R}$ **harmonic** in V, if $\forall x \in V$ and $r > 0$, if $B(x, r) \subset V$ then*

$$u(x) = \frac{\overbrace{\int \cdots \int}^{d}_{B(x,r)} u(y)dy}{\int \cdots \int_{B(x,r)} 1dy}.$$

In one dimension we get $u(x) = \frac{\int_{x-r}^{x+r} u(y)dy}{2r}$. All the Harmonic functions are linear in $\mathbb{R}$, i.e., $u(x) = ax + b, u'' = 0$.
In $\mathbb{R}^2 \setminus \{0\}$ $u(x) = \log |x|$ is harmonic.
In $\mathbb{R}^3 \setminus \{0\}$ $u(x) = |x|^{-1}$ is harmonic.
In general u harmonic $\Leftrightarrow \Delta u = \sum_i u_{x_i x_i} = 0$

Definition 7.7. *A point $x \in V$ is called a **local maximum** of $f : V \to \mathbb{R}$ if $\exists r > 0$ with $f(x) = \max\limits_{B(x,r)} f$*

Fact 7.8. *For a harmonic function u, if it has a local maximum at x then it is constant on some ball $B(x, r)$*

Suppose $V \subset \mathbb{R}^n$ is open and bounded, $u : \overline{V} \to \mathbb{R}$ continuous and whenever $u(x) = \max\{u(y) : y \in B(x, r)\}$ (where $B(x, r) \subset V$), then $u(x) = u(y)\ \forall y \in B(x, r)$
$-u$ has the same property: if $u(x) = \min\limits_{B(x,r)} u(y)$, then $u(y) = u(x)$ for all $B(x, r)$.

Claim 7.9. *If $u|_{\partial v} \equiv 0$ it follows that $u \equiv 0$ in all of $\overline{V}$.*

Proof. u has a local maximum on $\overline{V}, u(x_0) = \max u$. Our goal now is to show $u \leq 0$ on $\overline{V}$. So if $x_0 \in \partial V$ we are done. If $x_0 \in V$ then it is also a local maximum so $u(y) = u(x_0)$ $\forall y \in B(x_0, r)$ for some $r > 0$. Next suppose V is connected, Then $V = V_1 \cup V_2, V_1 = \{x \in V : u(x) = u(x_0)\}, V_2 = \{x \in V : u(x) < u(x_0)\} = u^{-1}\{(-\infty, u(x_0))\}$ These sets are both open. If V is connected this forces $V_2 = \emptyset$. If V is not connected, apply this to each connected component of V.

The same argument applied to $-u$ shows $-u \leq 0$ on $\overline{V} \Rightarrow u = 0$ on $\overline{V}$. □

CONNECTED COMPONENTS

path connected components. Say $x \sim y$ if there is a path from x to y. Then $\sim$ is an equivalence relation because it satisfies:
(i) $x \sim x$ Reflexive
(ii) $x \sim y \Leftrightarrow y \sim x$ Symmetric
(iii) If $x \sim y$ and $y \sim z$ then $x \sim z$ Transitive

Let $C_{path}(x) = \{y : y \sim x\}$, then it is path connected and has property (2).

Corollary 7.10. *For an open set $V \subset \mathbb{R}^n$ all the path connected components are open. In particular every open $V \subset \mathbb{R}^n$ can be written as a disjoint union of open intervals. We allow $(a, \infty), (-\infty, \infty), (-\infty, b), (a, b)$.*

Proof. $\forall x \in V$ we need to check C_{path} is open. $y \in C_{path} \Rightarrow \exists r > 0$ st $B(y, r) \subset V$. Then for any $z \in B(y, r)$, by taking a path from x to y and then taking a straight line path from y to z, we see that $B(y, r) \subset C_{path}$. □

Application: Suppose $K \subset \mathbb{R}$ is closed and $f : K \to \mathbb{R}$ is continuous, then $\exists \widetilde{f} : \mathbb{R} \to \mathbb{R}$ that is continuous and extends f. Extends means $\widetilde{f}|_K = f$

Proof. Let $K^c = \bigsqcup_{i=1}^{N} (a_i, b_i)$ where N could be ∞. If $a_i = -\infty$. Let $\widetilde{f}(x) = f(b_i)$ on $(-\infty, b_i)$ If $b_i = \infty$. Let $\widetilde{f}(x) = f(a_i)$ on (a_i, ∞) If $a_i, b_i \in \mathbb{R}$ and $a_i < x < b_i$ write $x = (1-t)a_i + tb_i$ for some $t \in (0, 1)$. Let $\widetilde{f}(x) = (1-t)f(a_i) + tf(b_i)$. Check $\widetilde{f}$ continuous at $x \in \mathbb{R}$. If $x \in K^c$ it is fine because linear functions are continuous. If $x \in K$, then $\forall \epsilon > 0$, $\exists \delta$ such that $f(B_k(x, \delta)) \subset B(f(x), \epsilon) \Rightarrow \widetilde{f}(B_{\mathbb{R}}(x, \delta)) \subset B(f(x), \epsilon)$. $a_i < y < b_i \Rightarrow f(a_i) < \widetilde{f}(y) < f(b_i)$ If a_i and b_i are both in a ball use continuity of linear functions to finish. □

Math H104: Honors Introduction to Analysis **Fall 2005**

Lecture 15: October 18

Lecturer: Yuval Peres *Scribe: Brian Shotwell*

Lemma 7.11. *Suppose $\{K_\alpha\}_{\alpha\in J}$ are connected sets in X, where $x \in K_\alpha$ for all $\alpha \in J$ for some point $x \in X$. Then $K = \bigcup_{\alpha\in K} K_\alpha$ is connected.*

Proof. Suppose $K \subset V_1 \cup V_2$, where V_i are open in X and disjoint in K (that is, $(V_1 \cap K) \cap (V_2 \cap K) = \emptyset$). Then $K = (K \cap V_1) \sqcup (K \cap V_2)$, a union of 2 disjoint sets that are open in K. We need to show $K \cap V_1$ or $K \cap V_2$ is empty.

Either $x \in V_1$ or $x \in V_2$. Without loss of generality, suppose that $x \in V_1$. Then for all α, $K_\alpha = (V_1 \cap K_\alpha) \sqcup (V_2 \cap K_\alpha) \implies V_2 \cap K_\alpha = \emptyset$ (since K_α is connected). Hence $\emptyset = \bigcup_{\alpha\in J}(V_2 \cap K_\alpha) = V_2 \cap K$. □

Definition 7.12. *Let X be a metric space and suppose $A \subset X$. For each $x \in A$ the connected component $C_A(x)$ of x in A is*

$$C_A(x) = \bigcup_{\alpha\in K} K_\alpha, \text{ where } x \in K_\alpha \subset A;\ K_\alpha \text{ connected.}$$

We claim the following facts about connected components:
a. $C_A(x)$ is connected.
b. $C_A(x)$ is the maximal connected set in A that contains x.
c. For all $x, y \in A$ either $C_A(x) = C_A(y)$ or $C_A(x) \cap C_A(y) = \emptyset$.

Proof. **a.** This is true by the above lemma.
b. If $D \subset A, x \in D$, and D connected, then $C_A(x) \supset D$ (by definition).
c. If $C_A(x) \cap C_A(y) = \emptyset$ we are done. Otherwise, there exists $z \in C_A(x) \cap C_A(y)$. By the lemma $C_A(x) \cup C_A(y)$ is connected. By the maximality of $C_A(x)$ and $C_A(y)$, $C_A(x) = C_A(x) \cup C_A(y) = C_A(y)$. □

Example 7.13. *A closed, connected set in $\mathbb{R}^2$ that is not path-connected:*

$$K = \left\{(x, \sin\left(\frac{1}{x}\right)) : 0 < x \le \frac{1}{\pi}\right\} \sqcup \{(0, y) : -1 \le y \le 1\} = G \sqcup L.$$

We claim that K is
(1) Closed.
(2) Connected.
(3) Not path-connected.

Proof. **1.** Suppose $(x_n, y_n) \to (x, y)$ and $(x_n, y_n) \in K$. We need to show $(x, y) \in K$. If $x = 0$, then $y \in [-1, 1]$ (and we are done). If $x \neq 0$ then by continuity of sin, it follows that if $\frac{1}{x_n} \to \frac{1}{x}$, then $\sin(\frac{1}{x_n}) \to \sin(\frac{1}{x})$. Hence, $y = sin(\frac{1}{x})$ and thus $(x, y) \in K$, and we are done.

2. $K = G \sqcup L$. Clearly G, L are connected (they're even path-connected). Suppose $K \subset$

$V_1 \cup V_2$, where V_1, V_2 are open in $\mathbb{R}^2$ and $V_1 \cap V_2 \cap K = \emptyset$. V_1 and V_2 cannot both intersect G, and V_1 and V_2 cannot both intersect L (because they are disjoint).

If both $V_1 \cap K$ and $V_2 \cap K$ are nonempty, one of them must be G and the other L. Without loss of generality, $G = V_1 \cap K$ and $L = V_2 \cap K$. $(0,0) \in V_2 \implies$ there exists an $\epsilon > 0$ such that $B((0,0),\epsilon) \subset V_2$. But $(\frac{1}{\pi n}, 0) \in G \supset V_1$ for all n. This gives a contradiction when $\frac{1}{\pi n} < \epsilon$ (as this implies that $(\frac{1}{\pi n}, 0) \in V_1 \cap V_2 \cap K$).

3. Proof by contradiction: suppose there is a path $\phi : [0,1] \to K$, satisfying $\phi(0) = (0,0)$, $\phi(1) = (\frac{1}{\pi}, 0)$, ϕ continuous.

Let $t_0 = \inf\{t : \phi_1(t) > 0\}$ where $\phi(t) = (\phi_1(t), \phi_2(t))$. Also, let $t_1 = \sup\{t : \phi_1(t) = 0\} = \sup \phi_1^{-1}(0) = \sup \phi^{-1}(L) = \max \phi^{-1}(L)$. $\phi_1(t_1) = 0$, $\phi(t_1) \in L$ (note we can replace the supremum with the maximum element because L is closed). We will choose to work with t_1 in the remainder of the proof, although one could use t_0.

By continuity of ϕ at t_1, there exists a $\delta > 0$ so that $\phi\left(B_{[0,1]}(t_1,\delta)\right) \subset B(\phi(t_1), 1/2)$. Either $\phi_2(t_1) \geq 0$ or $\phi_2(t_1) < 0$: both cases are similar.

Suppose $\phi_2(t_1) \geq 0$. Then $\phi_2(t_1, t_1 + \delta)$ does not contain -1. $\phi(t_1 + \delta/2)$ is connected by a path to $\phi(t_1) \in L$. Write $\phi(t_1 + \delta/2) = (x_1, y_1)$. Next, find a k with $0 < \frac{1}{(2k-1/2)\pi} < x_1$. We get a contradiction because $K \setminus \{(\frac{1}{(2k-1/2)\pi}, 0)\}$ is disconnected, a union of 2 relatively open sets and we have a path from one to the other.

If $\phi_2(t_1) < 0$, work with $\frac{1}{(2k+1/2)\pi}$, and the proof is complete.

□

Math H104: Honors Introduction to Analysis **Fall 2005**

Lecture 16: October 20

Lecturer: Yuval Peres *Scribe: Michael Gullans*

8. Outline

This document presents a transcription of the October 20th lecture. §9 contains the solutions for several problems from the practice midterm, while §10 presents the proof that any metric space is contained in a suitably, to be defined, unique, complete metric space.

9. Practice Midterm

Problem 9.1. *Show that* Card($\mathbb{R}$) = Card($\mathbb{R} \setminus \mathbb{Q}$).

The most direct way is to find an explicit bijection, without using the Schroeder-Bernstein theorem. Let D be a denumerable subset of $\mathbb{R} \setminus \mathbb{Q}$, for example $D = \mathbb{Q} + \sqrt{2}$ suffices because if there were a rational number in this set then $\sqrt{2}$ would be rational, being the sum of two rationals, an impossibility. We can enumerate $D \cup \mathbb{Q}$ with D because the union of two countable sets is countable, so by composing bijections with $\mathbb{N}$ we get a bijection $\phi : D \to D \cup \mathbb{Q}$. Now we can define a bijection $f : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$, by $f|_D = \phi$ and $f|_{\mathbb{R}\setminus(D\cup\mathbb{Q})} = \text{id}$. We see that f as the disjoint union of these two restrictions defines a bijective function because each restriction is a bijection. Thus the two sets in question are in bijective correspondence, implying that they have equal cardinality.

Problem 9.2. *Find a continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that there exists a closed set $A \subset \mathbb{R}^2$ with $f(A)$ not closed.*

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (e^x, e^y)$. This is a continuous function that maps $\mathbb{R}^2$ onto $(0, \infty)^2$. Now let $A = \mathbb{R}^2$, this is a closed set, but $(0, 0) \notin f(A)$ is a limit point of $f(A)$, so $f(A)$ is not closed.

Problem 9.3. *Show that* C[0, 1] *is connected, with the metric defined by the max norm.*

We show that C[0,1] is path-connected. Indeed it is convex: For $t \in [0, 1]$ and $f, g \in$ C[0, 1], the function $(1-t)f+tg$ is also in $C[0, 1]$. Thus $\gamma : [0, 1] \to$ C[0, 1] given by $\gamma(t) = (1-t)f+tg$, satisfies $\gamma(0) = f$ and $\gamma(1) = g$, i.e.,it is a path from f to g. (The simplicity of this proof may come as a frustration to those of you (such as the scribe) who proved directly that C[0,1] had no proper, clopen subsets).

Problem 9.4. *Find a continuous, bounded function $f : (0, 1) \to \mathbb{R}$, which is not uniformly continuous.*

Take the function $f : x \mapsto \sin(\frac{\pi}{x})$. $f((0, 1)) \subset B(0, 2)$, so f is bounded. Then for $\epsilon = 1/2$, assume there exists a δ satisfying the hypothesis of uniform continuity. Now find an even $n \in \mathbb{N}$ and an odd $m \in \mathbb{N}$, such that $|\frac{1}{n+\frac{1}{2}} - \frac{1}{m+\frac{1}{2}}| < \delta$, let $x_1 = \frac{1}{n+\frac{1}{2}}$ and $x_2 = \frac{1}{m+\frac{1}{2}}$. Such m, n obviously exist for any $\delta > 0$, but $|f(x_1) - f(x_2)| = 2 > \epsilon$, a contradiction. So f is not uniformly continuous despite being continuous on (0,1) (it is differentiable on (0,1)).

10. Metric Space completion

We first consider the question of when a continuous function $f : (0,1) \to \mathbb{R}$ can be extended to a continuous function on $\overline{(0,1)} = [0,1]$.

Proposition 10.1. *There is such an extension of f if and only if f is uniformly continuous.*

Proof. Because $[0,1]$ is compact, clearly uniform continuity is necessary. The reverse direction requires more work. We assume $f : (0,1) \to \mathbb{R}$ is uniformly continuous and construct a continuous extension $\widetilde{f} : [0,1] \to \mathbb{R}$, by $\widetilde{f}(1) = \lim_{n\to\infty} f(x_n)$, where (x_n) is a sequence converging to 1. Now we know that $\widetilde{f}(1)$ exists because f is uniformly continuous so $(f(x_n))$ is Cauchy in $\mathbb{R}$ and therefore converges, now we must show that the definition is well defined. Let (x_n), (y_n) be two sequences converging to 1, then the sequence (z_n), where $z_{2m} = x_m$ and $z_{2m-1} = y_m$, also converges to 1. Now, $(f(z_n))$ converges to some value, but it has two convergent subsequences $(f(x_n))$, $(f(y_n))$, so these must converge to the same value as the mother sequence. Thus $\lim f(x_n) = \lim f(y_n)$ and our definition is independent of the choice of sequence converging to 1. Now do the same thing for 0 and we have a continuous function on $[0,1]$, because $\widetilde{f}(\lim x_n) = \lim \widetilde{f}(x_n)$ for every convergent sequence (x_n) in $[0,1]$. □

Also we shall need the following definition.

Definition 10.2. *A map $\psi : X \to X'$, where X and X' are metric spaces with metrics d and d', respectively, is an* **isometry** *if $d'(\psi(x), \psi(y)) = d(x,y)$ for every $x,\ y \in X$.*

Clearly, such a map is (uniformly) continuous, just let $\delta = \epsilon$. With these facts in mind we proceed to the main theorem of this lecture. The only real difficulty in the following theorem is in discovering that it is true and stating its hypotheses and results rigorously.

Theorem 10.3 (Completion)**.** *Given a metric space (X,d), there always exists a complete metric space $(\widetilde{X}, \widetilde{d})$, where $X \subset \widetilde{X}$ and d(x,y)=$\widetilde{d}(x,y)$ for every $x,\ y \in X$, and $\widetilde{X} = \overline{X}$ in $\widetilde{X}$. Moreover, $\widetilde{X}$ is unique in the following sense: if (X^*, d^*) also has these properties, then there exists a surjective isometry $\psi : \widetilde{X} \to X^*$ such that $\psi|_X = \mathrm{id}$. Finally, given Y complete and $f : X \to Y$ uniformly continuous, then f extends to a continuous function $\widetilde{f} : \widetilde{X} \to Y$.*

Proof. Define $\widetilde{X_0}$ to be the set of equivalence classes of cauchy sequences in X, where $(x_n) \sim (y_n)$ if $d(x_n, y_n) \to 0$. Denote the elements of $\widetilde{X_0}$ as $[(x_n)]$, where (x_n) is a representative of an equivalence class. Now define $\widetilde{X}$ as follows:

$$\widetilde{X} = X \bigcup \{[(x_n)] \in \widetilde{X_0} : (x_n) \text{ is a non-converging Cauchy sequence in } X\}$$

Define $\widetilde{d_0}$ in $\widetilde{X_0}$ by letting $x = [(x_n)],\ y = [(y_n)] \in \widetilde{X} \setminus X$ and $\widetilde{d_0}(x,y) = \lim d(x_n, y_n)$. Such a metric is always defined because $\mathbb{R}$ is complete and it is well-defined by the same argument as in the previous proposition applied to two distinct representatives of the same equivalence class. We can define $\widetilde{d}$ on $\widetilde{X}$ by letting it equal the obvious metric when x and y are both in X, or both in $\widetilde{X_0}$ and $\widetilde{d}(x, [(y_n)]) = \lim d(x, y_n)$ when this is not the case. Similarly, this definition is valid and well-defined for every $x, y \in \widetilde{X}$. We have that $\overline{X} = \widetilde{X}$ in $\widetilde{X}$ because the set of accumulation points of X in $\widetilde{X}$ is just $\widetilde{X} \setminus X$ and the union of X and its accumulation points is $\overline{X}$.

Now we must check uniqueness of $\widetilde{X}$. Given X^*, define $\psi_0 : \widetilde{X_0} \to X^*$ by $\psi_0 : [(x_n)] \mapsto \lim x_n$ (Here lim denotes limit in X^*). This is an isometry by the definition of $\widetilde{X_0}$ and maps onto X^* because $\overline{X} = X^*$ in X^*. Also, it is clear that $\widetilde{X}$ and $\widetilde{X_0}$ are isometric, so to get an isometric surjection $\psi : \widetilde{X} \to X^*$ we can compose ψ_0 with this isometry.

Given a complete metric space Y and a uniformly continuous function $f : X \to Y$, we can define a continuous extension, $\widetilde{f}$, as follows: $\widetilde{f} : \widetilde{X} \to Y$, by $\widetilde{f} : [(x_n)] \mapsto \lim f(x_n)$ and $\widetilde{f}|_X = f$. These values always exist in Y because Y is complete and f is uniformly continuous, and it is well defined by the same argument as previous proposition. Also, $\widetilde{f}$ is continuous by the sequence definition of continuity. $\square$

Midterm: mathematics H104 - Oct 25, 2005

INSTRUCTOR: Yuval Peres DURATION: 75 minutes.

Instructions: Please write your name on **every** page. This examination contains four problems with weight 34 points each. Solve three of them. Write each answer **very clearly** below the corresponding question. (Use back of page if needed). **Good Luck**!

(1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the sum function, $f(x, y) = x + y$. For each of the following statements, provide a counterexample if it is false and a proof if it is true.
  (a) If $A \subset \mathbb{R}^2$ is complete then so is $f(A)$.
  (b) If $A \subset \mathbb{R}^2$ is connected then so is $f(A)$.
  (c) If $A \subset \mathbb{R}^2$ is open in $\mathbb{R}^2$ then $f(A)$ is open in $\mathbb{R}$.
  (d) If $E \subset \mathbb{R}$ is complete then $f^{-1}(E)$ is complete.
  (e) (*) If $E \subset \mathbb{R}$ is connected then $f^{-1}(E)$ is connected.

(2) Let $f : [0, 1] \to \mathbb{R}$ be a function, and let $G_f = \{(x, f(x)) : x \in [0, 1]\}$ be its graph.
  (a) Show that if f is continuous then G_f is closed in $\mathbb{R}^2$.
  (b) Give an example of a discontinuous function $f : [0, 1] \to \mathbb{R}$ such that its graph is closed in $\mathbb{R}^2$.
  (c) Does there exist an example f as in part (b) that is also bounded? (If so, provide one.)

(3) Show that if Y is a sequentially compact subset of a metric space X then Y is closed in X.

(4) Let X be an infinite, connected metric space. Show that X is not countable.

Math H104: Honors Introduction to Analysis **Fall 2005**

Lecture 17: October 27

Lecturer: Yuval Peres *Scribe: Jacob Porter*

This lecture is devoted to midterm solutions.

(1) For $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = x + y$ prove whether or not the following are true or false.
  (a) A complete and $A \subset \mathbb{R}^2 \implies f(A)$ is complete.
  (b) A connected $\implies f(A)$ connected.
  (c) A open $\implies f(A)$ open.
  (d) E complete and $E \subset \mathbb{R} \implies f^{-1}(E)$ complete.
  (e) **Bonus points:** E connected and $E \subset \mathbb{R} \implies f^{-1}(E)$ connected.

Solution

  (a) False. Let $A = \{(x, y)|x \leq 0, x^2 - y^2 = 1\}$. The set A is closed, but $0 \notin f(A)$. For $x + y = r$, $x - y = \frac{1}{r}$, $x = \frac{r+\frac{1}{r}}{2}$, and $y = \frac{r-\frac{1}{r}}{2}$, any positive r is in $f(A)$.
  (b) True. Theorem 2.45 says that for every continuous function this is true.
  (c) True. Suppose $r \in f(A)$, so $r = x+y$ with $x, y \in A$. Is there some $\epsilon > 0$ such that $(r-\epsilon, r+\epsilon) \subset f(A)$? We know there exists $\delta > 0$ such that $B((x, y), \delta) \subset A$, and $(r - \delta, r + \delta) \subset f(B((x, y), \delta))$. Why? Consider some point $r + \alpha \in (r - \delta, r + \delta)$, then $|\alpha| < \delta$, $f(x + \frac{\alpha}{2}, y + \frac{\alpha}{2}) = r + \alpha$, and $\|(x + \frac{\alpha}{2}, y + \frac{\alpha}{2}) - (x, y)\|_2 = \frac{\alpha}{\sqrt{2}} < \delta$. Thus, $(r - \delta, r + \delta) \subset f(A)$.
  (d) True. In $\mathbb{R}$ and $\mathbb{R}^2$ this is the same as if the set is closed. Then note that by continuity of f, $f^{-1}(V)$ is opn for any open V. Since $f^{-1}(V)^c = (f^{-1}(V))^c$ the same holds for closed sets.
  (e) True. Suppose $f^{-1}(E) \subset V_1 \cup V_2$ and V_1, V_2 open and $V_1 \cap V_2 = \emptyset$. Then $E \subset f(V_1) \cup f(V_2)$ because f is onto $\mathbb{R}$. Check that $f(V_1) \cap f(V_2) \cap E = \emptyset$ and then we are done. Suppose not. Then there exists $r \in f(V_1) \cap f(V_2) \cap E$. This is impossible because then $f^{-1}(r) = \{(x, y) \in \mathbb{R}^2 : x+y = r\}$ with $f^{-1}(r) \cap V_1 \cap V_2 \neq \emptyset$, but a line is path-connected; hence connected, a contradiction.

(2) For $f : [0, 1] \to \mathbb{R}$ and its graph $G_f = \{(x, y) \in f\}$ do the following:
  (a) If f is continuous does this imply that the graph G_f is closed?
  (b) Find an example of f that is discontinuous and that the graph G_f is closed.
  (c) Is there an example as in (b) that is also bounded? If not why not? If so, provide an example.

Solution

  (a) True. $G_f \ni (x_n, y_n) \to (x, y)$. By continuity, $y_n = f(x_n) \to f(x)$. $d(y, f(x)) \leq d(y, y_n) + d(y_n, f(x)) \to 0$. So, $0 = d(y, f(x))$, and $y = f(x)$.

(b) $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 5 & x = 0 \end{cases}$
If $x_n \nrightarrow 0$ then $(x, y) \in G_f$. If $x_n \to 0$ then $y_n \to \infty$, a contradiction. If $x_n = 0$ then for some point, $y_n = 5$.

(c) No. Suppose f is bounded, and G_f is closed. Check continuity. $x_n \to x$. Need $f(x_n) \to f(x)$. If not there exists $\epsilon > 0$ such that $|f(x_{n_k}) - f(x)| \geq \epsilon$ and n_k is increasing. $(x_{n_k}, y_{n_k}) \in G_f$, which is compact because G_f is closed and bounded. Thus, there exists k_j increasing such that $(x_{n_{k_j}}, y_{n_{k_j}}) \to (x_*, y_*) \in G_f$. $x_* = x$ and $y_* = f(x)$. However, $y_{n_{k_j}} = f(x_{n_{k_j}})$, and $|y_{n_{k_j}} - f(x)| \geq \epsilon$. This is a contradiction because $y_{n_{k_j}}$ should be converging to $f(x)$. (It is typical of proofs like this with compact sets to show this by contradiction).

(3) Prove that if $Y \subset X$ and if Y is sequentially compact then Y is closed in X.

Solution Suppose $(y_n) \in Y$ and $(y_n) \to z$. Need to show that $z \in Y$. By sequential compactness there exists n_k increasing such that $y_{n_k} \to y \in Y$, but since (y_{n_k}) is a subsequence, then $(y_{n_k}) \to z$ and $z = y$. Thus, $z \in Y$ as required.

(4) X is an infinite and connected metric space. Show X uncountable.

Solution The proof is by contradiction. Suppose X is countable. $X = \{x_j\}_{j=1}^{\infty}$. Goal: Find $r > 0$ with $B(x_1, r) = \bar{B}(x_1, r) = \{z \in X : d(z, x_1) \leq r\}$. Any $r \notin \{d(x_1, x_j)\}$ will do provided that $r > 0$ and $r < d(x_1, x_2)$.

Second Midterm: math H104 - Nov 8, 2005

INSTRUCTOR: Yuval Peres DURATION: 75 minutes.

Instructions: Please write your name on **every** page. This examination contains four problems with weight 34 points each. Solve three of them. Write each answer **very clearly** below the corresponding question. (Use back of page if needed). **Good Luck**!

(1) Prove that $[0, 1]$ is uncountable.

(2) Let (X, d) be a metric space.
  (a) Define what it means for X to be **totally bounded.**
  (b) Is the open interval $(0, 1)$, with the usual metric, totally bounded? Prove your answer from the definition without using any theorems.
  (c) Is there a metric space (X, d) where $d(x, y) < 1$ for all $x, y \in X$ yet X is not totally bounded ? Explain your answer.

(3) For each of the following statements, determine if it is true or false, and explain briefly.
  (a) If X is a finite nonempty metric space, then all subsets of X are clopen sets in X.
  (b) If X is a countable nonempty metric space, then there exists some $x \in X$ such that the one-point set $\{x\}$ is clopen in X.
  (c) If an open set V in $\mathbb{R}$ contains all the rationals, then $V = \mathbb{R}$.

(d) If K is a closed uncountable set in $\mathbb{R}$, then there exist $a < b$ in $\mathbb{R}$ such that $[a, b] \subset K$.

(4) Let (X, d) and (Y, ρ) be metric spaces. Suppose that the function $f : X \to Y$ is continuous onto Y and that X is compact. Prove that Y is also compact (Use the covering definition).